Aspects of noncommutative spectral geometry

Watcharangkool, Apimook

Awarding institution:
King's College London

Aspects of noncommutative spectral geometry

KING'S COLLEGE LONDON

DOCTORAL THESIS

Author:
Apimook Watcharangkool

Supervisor:
Prof. Mairi Sakellariadou

A thesis summitted in fulfilment of the requirements for the degree of Doctor of Philosophy in the Theoretical Particle Physics and Cosmology Group Department of Physics

March 20, 2017

Abstract

This thesis presents aspects of noncommutative spectral geometry as an approach to formulate a model of gravity and particle physics, while addressing open issues associated with this approach. We propose a novel definition of the bosonic spectral action using zeta function regularisation, in order to address the issues of renormalisability, ultraviolet completeness and spectral dimensions. We compare the zeta spectral action with the usual (cutoff based) spectral action and discuss its purely spectral origin, predictive power, stressing the importance of the issue of the three dimensionful fundamental constants, namely the cosmological constant, the Higgs vacuum expectation value, and the gravitational constant. We emphasise the fundamental role of the neutrino Majorana mass term for the structure of the bosonic action. We subsequently show that the regularised zeta spectral action gives a stable linearised gravitational theory despite being a 4th-order derivative theory. Afterwards, we explore the notion of Lorentzian noncommutative geometry, where the bosonic action is not well-defined. However, in such a case, the dynamics of fermions is still well-defined. We have shown that one could give a geometrical meaning to the energy-momentum dispersion relation of fermions.

Acknowledgements

First of all I would like to thank my supervisor Mairi Sakellariadou, for her patience, kindness, and most of all her valuable advise. I know you are a busy person, and I am sorry for being such a troublesome student. During my PhD I met many mathematicians and physicists who kindly offer help, advise, opportunity, and sometimes entertainment. These people are including Walter van Suijlekom, Michal Eckstein, Prof. Yuri Safarov (who passed away a few years ago), Prof. Fedele Lizzi, Max Kurkov, Bence Hovart, Tanapoom Poomaradee (entertainment), Nipol Chaemchumrus, Chakrit Pongkitivanichkul, Marco de Cesare, Julio Leite etc. I also owes a special thank (and a big meal) to Dr. Paolo Bertozzini for his very helpful comments and advise. This thesis cannot be finished without funding, which is provided by the Royal Thai Government scholarship.

Contents

Chapter 1

Introduction

The study of noncommutative geometry (NCG) was inspired by the idea that spacetime will reveal its quantum nature at some very high scales; the coordinate algebra of spacetime will be replaced by a noncommutative operator algebra. Such an idea leads to several formulations of noncommutative geometry; in this thesis we focus on the noncommutative spectral geometry (NCSG) approach. The notion of spectral geometry is inspired by the duality between a commutative C^*-algebra and a topological space i.e. the commutative algebra is nothing but the algebra of functions living on some topological space. Since any Riemannian manifold is a topological space with a differential structure, it is shown by Alain Connes [1] that by adding a Dirac operator together with some axioms, one can reconstruct a Riemannian spin manifold from the commutative pre$-C^*$-algebra. When the commutative algebra is replaced by a noncommutative one, the duality leads to the discovery of a new kind of geometrical objects such as noncommutative tori [2], noncommutative spheres, and almost commutative manifolds, which will have a central role in this thesis.

Physics from almost commutative geometry

An almost commutative manifold (AC manifold) is, roughly speaking, a manifold whose coordinate algebra is given by a matrix algebra. The geometrical structure of the AC manifold is equivalent to a product between a Riemannian manifold M, and an internal space F. In the case that the matrix algebra is the direct sum of smaller matrix algebras, the internal space is the disjoint union of topological spaces. Ignoring the structure of each internal space, one may think of $M \times F$ as a disjoint union of Riemannian manifolds. However, the distance between these disjoint manifolds can be either finite or infinite.

Although the dimension of $M \times F$ and that of the Riemannian manifold M are equal, the AC manifold has the larger symmetry group. The symmetry group of the AC manifold is, roughly speaking, the product of the Diffeomorphism group of M and the symmetry group of the internal space. Choosing an appropriate AC manifold such that its symmetry group contains the gauge group of the Standard Model (SM)[3, 4] leads to a model of particle physics coupling to gravity. In this model, gauge fields and scalar fields (which will become SM Higgs field) are generated by the fluctuation of the metric, and the four forces are given by the curvature induced by the symmetry of the AC manifold. Hence, one can say that the almost commutative geometry approach offers a fundamental description of SM as a gravitational theory on an AC manifold.

NCSG provides not only a fundamental picture of SM, but also phenomenological predictions. The mass of the Higgs boson was predicted to have the same order of magnitude to the experimental value. It was suggested that the inaccuracy of the prediction may stem from the big desert hypothesis [4]. By introducing a new scalar field [5], which does not exist in the original approach, the experimental value of the standard model Higgs mass can be obtained. The inclusion of this scalar field into the model has been realised by various approaches [6, 7, 8, 9].

A hint towards quantum gravity

A goal of NCSG is to construct a geometrical space associated with a certain operator algebra, therefore, given a quantum algebra, e.g. the Heisenberg algebra, deformed Hopf algebras (quantum groups), one would expect a quantum generalisation of Riemannian manifold. The Heisenberg commutation relation was generalised into the language of NCSG in Ref. [10]. It was shown that the spectral geometry satisfying the generalised commutation relation leads to the decomposition of a Riemannian manifold into disconnected spheres of unit volume (Planck units). Such a granular structure is a common feature one finds in quantum spacetime associated with many theories of quantum gravity. This result seems to suggest an approach towards quantum gravity.

Since there are various approaches to quantum gravity, one may ask: what are benefits of presenting a quantum gravity theory in the language of NCSG. The first benefit is that, in principle, the ambiguity of passing from a classical theory to a quantum one can be eliminated. Traditionally, the construction of a quantum gravity theory begins by choosing a gravitational action on a Riemannian manifold (pseudo-Riemannian manifold), then quantising the theory. However, NCSG provides an idea that one may first choose a quantum (noncommutative) algebra, then construct a quantum spacetime, and then define a theory of quantum gravity, hence, eliminates the need of quantisation. The research in this direction has been pursued in Ref. [11, 12, 13]. The second benefit is that NCSG offers a candidate for the line element in quantum gravity [14]; there is a natural notion of line element given by the inverse of the Dirac operator (A brief discussion on the distance function in NCSG can be found in section 2.2).

About this thesis

Applications of NCSG can be found in both low energy phenomena such as the quantum Hall effect [15, 16], and high energy ones such as physics of the Standard Model. In this thesis, our main interest lies within the area of high energy physics. Although NCSG has promising potential as a fundamental theory of gravity and particle physics, so far this approach can only provide an effective theory, due to various open issues. The aim of this thesis is to improve this potential by addressing some of those open issues.

This thesis is based on three publications [17, 18, 19]. The contents of the thesis are organised as follows. We start with some necessary elements of noncommutative spectral geometry in Chapter 2, including C^*-algebra, distance formula in NCSG, inner perturbations, and spectral invariants. In Chapter 3, we discuss various open issues associated to the conventional spectral action such as locality, renormalisability, hierarachy problem, and then we introduce another spectral action, based on the spectral zeta function, which partially addresses these issues. Then in Chapter 4, we discuss the gravitational part of the spectral action, and show that one could avoid negative energy that is a generic problem in higher derivative gravitational theories. In Chapter 5, we explore the origin of energy momentum dispersion relation in the context of Lorentzian spectral geometry. Finally we summarise our results and discuss some open questions and future research directions which arise from the investigations presented in this thesis.

Chapter 2

Elements of spectral geometry

Spectral geometry refers to the idea that one can construct a geometrical space from the spectral data of operators. This idea originates from the theorem proposed by Gelfand and Naimark in 1943, which offers a duality between commutative operator algebras and topological spaces. Later the duality was developed such that it enables one to construct a Riemannian spin manifold from a commutative operator algebra, therefore, leads to the well-defined notion of spectral geometry. Since operator algebra can be noncommutative, one may ask, what kind of geometry would arise when replacing the commutative operator algebra with a noncommutative one. A partial answer based on existing examples is that one obtains a generalised geometrical structure that retains some of the features of Riemannian spin geometry such as a metric, a spin structure, while it offers new features such as an internal symmetry and inner fluctuations (this will be discussed in more details later on). Such geometry is called noncommutative spectral geometry (NCSG) which is a type of noncommutative geometry.

The elements of spectral geometry, which we will use in the subsequent Chapters, can be summarised into the following three topics; elements of spectral triples, geometrical structures, and spectral invariants. In section 2.1, we first highlight the motivation behind the notion of spectral geometry, which then leads to the definition of spectral triple that has become the standard notion of spectral geometry. In section 2.2 we introduce the notion of distance defined purely from the spectral data. In section 2.3 we discuss the definition of symmetry group, its role in spectral geometry, and how it gives rise to gauge fields. The last section concerns spectral invariants of the Dirac operator such as heat kernel trace, zeta function and index. We then define the dynamics on spectral geometry based on these spectral invariants.

2.1 From C^*-algebra to geometry

As mentioned above, the notion of spectral geometry is developed from the duality between a commutative C^*-algebra and a topological space. A C^*-algebra is defined as follows:

Definition 2.1.1. *Let $\mathcal{A}$ be an algebra over the field of complex numbers $\mathbb{C}$ with a norm $\|\cdot\| : \mathcal{A} \to \mathbb{C}$. $\mathcal{A}$ is a **Banach algebra** if it is complete under the norm and satisfies*

$$\|ab\| \leq \|a\|\|b\| \quad \forall a, b \in \mathcal{A} \ . \tag{2.1.1}$$

*A **C*-algebra** is a Banach algebra equipped with a conjugate linear map $* : \mathcal{A} \to \mathcal{A}$ with properties $(a^*)^* = a$, and $(ab)^* = b^* a^*$, satisfying the equality*

$$\|a^* a\| = \|a\|^2. \tag{2.1.2}$$

Example 2.1.2. $\mathbb{C}$ *is a* C^**-algebra, for* $z \in \mathbb{C}$ *the star operation and norm are defined by* $z^* := \bar{z}$ *and* $\|z\| := |z|$.

Example 2.1.3. *The algebra* $\mathcal{B}(\mathcal{H})$ *of bounded linear operators on a Hilbert space* $\mathcal{H}$ *is a* C^**-algebra. Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $x \in \mathcal{H}$*, the operator norm is defined by*

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \ , \tag{2.1.3}$$

and the $*$*-map is defined by the adjoint operation. In particular, for* $\mathcal{H} = \mathbb{C}^n$ *one can deduce that* $M_n(\mathbb{C})$ *is also a* C^**-algebra.*

Example 2.1.4. *An operator* T *is compact if for any bounded subset* $U \subset \mathcal{H}$*, the closure of* $T(U)$ *is compact. The collection of such operators, denoted by* $\mathcal{K}(\mathcal{H})$*, is a* C^**-algebra, and also a closed subalgebra of* $\mathcal{B}(\mathcal{H})$*.*

Actually any C^*-algebra $\mathcal{A}$ is isomorphic to a closed subalgebra of $\mathcal{B}(\mathcal{H})$. The connection between C^*-algebras and operator algebras leads to the notion of the spectrum of an element of a C^*-algebra. For a unital C^*-algebra, the spectrum of an element $a \in \mathcal{A}$ is defined by

$$\mathrm{Sp}(a) = \{\lambda \in \mathbb{C} \ ; \ a - \lambda 1 \notin \mathcal{A}^\times\} \ , \tag{2.1.4}$$

where $\mathcal{A}^\times$ denotes the group of invertible elements. The definition of spectrum Eq. (2.1.4) is applicable to a nonunital C^*-algebra, providing that one unitises (adding a unit) the algebra. A **unitisation** can be achieved by embedding a nonunital C^*-algebra $\mathcal{A}$ in a larger C^*-algebra $\tilde{\mathcal{A}} = \mathcal{A} \times \mathbb{C}$ with the product defined by

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu) \ , \tag{2.1.5}$$

and the norm

$$\|(a,\lambda)\| = \sup\{\|ab + \lambda b\| \ ; \ \|b\| \leq 1\} \ . \tag{2.1.6}$$

Next we will define the notion of character, which plays a crucial role in the Gelfand-Naimark theorem that we will discuss later.

Definition 2.1.5. *A* **character** *is a nonzero surjective homomorphism $\mu : \mathcal{A} \to \mathbb{C}$. The collection of all characters is denoted by $M(\mathcal{A})$.*

One can show that $\mu(a) \in \mathrm{Sp}(a)$. Assuming $\mu(a) \notin \mathrm{Sp}(a)$, so that $a - \mu(a)1$ is invertible, then

$$\mu(a - \mu(a)1) = \mu(a) - \mu(a) = 0 \ . \tag{2.1.7}$$

Since μ is a homomorphism, it maps an invertible element to another one. This leads to a contradiction since 0 is not invertible. If $\mathcal{A}$ is commutative, then we call $M(\mathcal{A})$ the **Gelfand spectrum**. In the case of $\mathcal{A} = C(X)$ is the algebra of continuous functions on a compact Hausdorff space X, the Gelfand spectrum is the collection of evaluation maps $\epsilon_x : f \to f(x)$ at $x \in X$. It is obvious that the map from $\hat{\varphi} : M(C(X)) \to X$ defined by $\epsilon_x \mapsto x$ is a bijection. By imposing the weak* topology on $M(C(X))$, the map $\hat{\varphi}$ is a homeomorphism [20].

Lemma 2.1.6. *The Gelfand spectrum of a unital commutative C^*-algebra is a compact Hausdorff space.*

In the case of nonunital algebra $\mathcal{A}$, one applies Lemma 2.1.6 to $M(\tilde{\mathcal{A}})$ (recall that $\tilde{\mathcal{A}}$ is the unitisation of $\mathcal{A}$), and then extracts the topological structure of $M(\mathcal{A})$. Let $(a,\lambda) \in \tilde{\mathcal{A}}$ and $\mu \in M(\mathcal{A})$, characters on $\tilde{\mathcal{A}}$ given by

$$\tilde{\mu}(a,\lambda) = \mu(a) + \lambda \quad \text{and} \quad \mu_0(a,\lambda) = \lambda \ , \tag{2.1.8}$$

therefore,

$$M(\tilde{\mathcal{A}}) = M(\mathcal{A}) \cup \{\mu_0\} \ . \tag{2.1.9}$$

Since $M(\tilde{\mathcal{A}})$ is compact, the space $M(\mathcal{A}) \cup \{\mu_0\}$ is also compact. By removing the point $\{\mu_0\}$, we have that $M(\mathcal{A})$ is a locally compact (Hausdorff) space.

Alternatively, one may identify a point on a space with a pure state of a commutative C^*-algebra, defined as follows.

Definition 2.1.7. *The set of* **states** $\mathcal{S}(\mathcal{A})$ *is the collection of positive linear functionals of norm one. The set* $\mathcal{S}(\mathcal{A})$ *is convex, and its extreme points (any state that cannot be written as convex combination of other states) are called* **pure states**. *The set of pure states will be denoted by* $\mathcal{P}(\mathcal{A})$.

Although the sets $M(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ are the same in the commutative case, since their elements are given by the evaluation map ϵ_x, they are not necessary equal in the noncommutative case. In mathematical physics (and also in this thesis), the generalised notion of point on a noncommutative geometry is often defined as pure states, instead of Gelfand characters.

Theorem 2.1.8. *(Gelfand-Naimark Theorem)*
Let $\mathcal{A}$ *be a unital commutative* C^**-algebra, there exists an isometric* $*$*-isomorphism such that* $\mathcal{A} \cong C(M(\mathcal{A}))$

If the algebra is not unital, then it is isomorphic to the algebra of continuous functions vanishing at infinity denoted by $C_0(M(\mathcal{A}))$. One should notice that the Gelfand-Naimark theorem and the lemma 2.1.6 imply that the compactification of a locally compact Hausdorff space is equivalent to the unitisation of a nonunital C^*-algebra

$$C(M(\tilde{\mathcal{A}})) = C(M(\mathcal{A}) \cup \{\mu_0\}) = C_0(M(\mathcal{A})) \oplus \mathbb{C} \ . \tag{2.1.10}$$

Hence, we have a generalised notion of compactness in the language of C^*-algebra.

One deduces from the Gelfand-Naimark theorem that any unital commutative C^*-algebra can be realised as an algebra of continuous functions on some compact Hausdorff space (and a locally compact Hausdorff space for a nonunital one). Since a compact Hausdorff space is the basic ingredient for any smooth compact manifold, one may ask what kind of extra ingredients are needed in order to upgrade $M(\mathcal{A})$ from a topological space to a manifold. An answer was first suggested, and subsequently proven by Alain Connes [21, 1] showing that one can obtain a manifold by introducing the notion of spectral triples.

Definition 2.1.9. *Spectral Triple*

A spectral triple is a collection of data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}$ is a C^*-algebra (or pre-C^*-algebra, which is not necessary complete), $\mathcal{H}$ is a Hilbert space carrying a representation of $\mathcal{A}$ as bounded operators, and $\mathcal{D}$ is a Dirac operator i.e. a densely defined self-adjoint operator such that

- *the resolvent of $\mathcal{D}$ is a compact operator.*
- *for each $a \in \mathcal{A}$, the commutator $[\mathcal{D}, a]$ is a bounded operator on $\mathcal{H}$.*

Remark. *Having a compact resolvent, which means that for $\lambda \notin \mathrm{Sp}(\mathcal{D})$ the resolvent operator $R(\lambda, \mathcal{D}) := (\mathcal{D} - \lambda 1)^{-1}$ is compact, implies that $\mathcal{D}$ has discrete spectrum* [22].

A spectral triple is **even** if there exists a grading operator γ such that

$$\gamma a = a\gamma, \quad \text{and} \quad \gamma\mathcal{D} = -\mathcal{D}\gamma, \tag{2.1.11}$$

and it is **real** if there exists an anti-linear isometry $J : \mathcal{H} \to \mathcal{H}$ such that

$$J^2 = \epsilon, \quad J\mathcal{D} = \epsilon'\mathcal{D}J, \quad J\gamma = \epsilon''\gamma J \text{ (even case)}, \tag{2.1.12}$$

where $\epsilon, \epsilon', \epsilon'' \in \{-1, 1\}$. There are only 8 possible KO-dimension of a real spectral triple [21, 4]. The justification of the axioms (2.1.12) are in the analogy with the periodicity modulo 8 in Riemannian spin geometry.

Table 2.1: KO-dimension of Riemannian spin manifolds

n mod 8	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Suppose $\mathcal{A}$ acting on $\mathcal{H}$ by a left action, the existence of real structure allows one to define the right action $ha^0 = Ja^*J^*h, \quad \text{for } h \in \mathcal{H}$. The right action needs to satisfy two additional axioms

$$[a^0, b] = 0 \tag{2.1.13}$$

$$[a^0, \mathcal{D}b] = 0, \tag{2.1.14}$$

which are often called zero-order and first-order axioms. Both axioms are necessary for the reconstruction of a Riemannian spin manifold, which we will shortly discuss. Note that, there is an example of real spectral triple in which the first-order axiom is relaxed [23].

A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ can be constructed from a compact four-dimensional Riemannian spin manifold with a spinor bundle $S \to M$ in the following way.

- $\mathcal{A}$ is given by the set of smooth, infinitely differentiable functions $C^\infty(M)$ with pointwise multiplication.
- $\mathcal{H} := L^2(M, S)$ is the Hilbert space of square-integrable spinors on M.
- the Dirac operator $\mathcal{D} := -i\gamma^\mu \nabla^S_\mu$ is defined in terms of the spin Levi-Cività connection ∇^S and the Dirac gamma matrices γ^μ.
- the grading is given by $\gamma^5 := \gamma^0\gamma^1\gamma^2\gamma^3$
- the real structure is $J_M := \gamma^0\gamma^2 \circ cc$ (also known as the charge conjugation operator), where cc denotes complex conjugation.

Such spectral triple is called the **canonical triple**. Conversely, given a commutative spectral triple, one can reconstruct a four-dimensional Riemannian manifold. Note that, this result holds for any even-dimensional manifold but for the purpose of this thesis we often assume manifolds to be four-dimensional, unless state otherwise.

Theorem 2.1.10. *(Manifold reconstruction theorem* [1]*)*
Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J)$ be a commutative even graded real spectral triple satisfying the five conditions given in Ref. [1]*. There exists a compact (even dimensional) Riemannian spin manifold M such that $\mathcal{A} \cong C^\infty(M)$.*

For a locally compact manifold, the algebra is replaced by the algebra of smooth functions vanishing at infinity $C_0^\infty(M)$, which is a nonunital algebra.

The manifold reconstruction theorem shows that spectral data can characterise the geometry of ordinary Riemannian manifolds, in the sense that the canonical spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ encodes the manifold structure. Then, if the definition of spectral triple is regarded as a notion of generalised geometry, one would expect a new kind of geometry to emerge when $\mathcal{A}$ is a noncommutative algebra: such geometry is dubbed **noncommutative spectral geometry** (NCSG).

The study of noncommutative spectral geometry is directly connected to a fundamental model of particle physics. The essential properties of a particle physics model such as symmetry, particle content, and mass matrices can be nicely encoded in the language of spectral triple. Inspiring by the local gauge group of the Standard Model (SM)

$$G_{\text{SM}} = U(1) \times SU(2) \times SU(3) \ ,$$

one may choose an algebra that contains such a gauge group, e.g.

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \ . \tag{2.1.15}$$

The fermions of SM are represented by a finite-dimensional Hilbert space $\mathcal{H}_F = (\mathbb{C}^{32})^{\oplus 3}$, where each $\mathbb{C}^{32}$ is divided into

$$\mathcal{H}_l \oplus \mathcal{H}_q \oplus \mathcal{H}_{\bar{l}} \oplus \mathcal{H}_{\bar{q}} = \mathbb{C}^4 \oplus (\mathbb{C}^4 \otimes \mathbb{C}^3) \oplus \mathbb{C}^4 \oplus (\mathbb{C}^4 \otimes \mathbb{C}^3) \ . \tag{2.1.16}$$

Each $(\mathcal{H}_l \oplus \mathcal{H}_q \oplus \mathcal{H}_{\bar{l}} \oplus \mathcal{H}_{\bar{q}})^i$ represents fermions in the generator i-th, $i = 1, 2, 3$, with the basis $\{v(i)_R,\ v(i)_L,\ \bar{v}(i)_R,\ \bar{v}(i)_L\}$, for $v(i) = \nu_i,\ e_i,\ u_i^c,\ d_i^c$ where $c = r, g, b$ is the colour index. The representation of $a = (\lambda, q, m) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ on the

Hilbert space is defined by

$$\pi_{\mathcal{H}_l}(a) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix} , \qquad \pi_{\mathcal{H}_q}(a) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix} \otimes \mathbb{1}_3$$

$$\pi_{\mathcal{H}_{\bar{l}}}(a) = \lambda \mathbb{1}_4 , \qquad \pi_{\mathcal{H}_{\bar{q}}}(a) = \mathbb{1}_4 \otimes m,$$

where $\alpha, \beta \in \mathbb{C}$. The Dirac operator is a self-adjoint operator $\mathcal{D}_F$ that encodes the mass matrices of fermions

$$\mathcal{D}_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix} . \tag{2.1.17}$$

Here $T(\nu_i)_R = (Y_R)_{ik}(\overline{\nu}_k)_R$, where Y_R is the 3×3 Majorana mass matrix, and $Tv = 0$ on any other basis of $(\mathcal{H}_l \oplus \mathcal{H}_q)^i$, and

$$S|_{\mathcal{H}_l^{\oplus 3}} = \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ Y_\nu^* & 0 & 0 & 0 \\ 0 & Y_e^* & 0 & 0 \end{pmatrix} , \quad S|_{\mathcal{H}_q^{\oplus 3}} \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ Y_u^* & 0 & 0 & 0 \\ 0 & Y_d^* & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_3 ,$$

where Y_ν, Y_e, Y_u and Y_d are the 3×3 Yukawa mass matrices. The spectral triple

$$(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F) , \tag{2.1.18}$$

is called the finite spectral triple. The grading and the real structure are given by

$$\gamma_F v(i)_L = v(i)_L, \quad \gamma_F v(i)_R = -v(i)_R \ ,$$
$$\gamma_F \bar{v}(i)_R = \bar{v}(i)_R, \quad \gamma_F \bar{v}(i)_L = -\bar{v}(i)_L \ , \tag{2.1.19}$$

and

$$J_F v(i)_{R,\ L} = \bar{v}(i)_{R,\ L} \ , \quad \text{and} \quad J_F \bar{v}(i)_{R,\ L} = v(i)_{R,\ L} \ . \tag{2.1.20}$$

Hence we have a real even spectral triple of KO-dimension 6. From this example, one should not conclude that the choice of algebra $\mathcal{A}_F$ is arbitrary as long as it contains G_{SM}. As it was shown in the Ref. [24], the finite spectral triple that (i) contains the algebra (2.1.15), (ii) gives the particle content of SM must have KO-dimension 6 and its algebra must be of the form

$$M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C}), \ a \in \mathbb{N} \ . \tag{2.1.21}$$

For $a \geq 2$. For the case $a = 2$, the spectral triple with the zero-order and first-order axioms yields exactly the algebra (2.1.15). Note that, Choosing a larger algebra or relaxing the first-order axiom may add a new particle into the model as in Ref. [6, 7, 8, 25].

We have learned from the commutative case that the pure states of the algebra are equivalent to points on a topological space. By generalising this idea one finds that the spectral triple $(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F)$ describes a finite set of disjoint topological space, since $\mathcal{P}(\mathcal{A}_F) = \coprod_i \mathcal{P}(\mathcal{A}_i) =: F$. Although the finite spectral triple by itself gives a trivial structure, its product with the canonical spectral triple, namely

$$(C_0^\infty(M) \otimes \mathcal{A}_F, L^2(M,S) \otimes \mathcal{H}_F, -i\nabla\!\!\!\!/ \otimes \mathrm{Id}_F + \gamma^5 \otimes \mathcal{D}_F) \ , \tag{2.1.22}$$

together with the grading operator $\gamma^5 \otimes \gamma_F$ and the real structure $J_M \otimes J_F$, yields a nontrivial noncommutative structure [4] (note that the tensor product of two real spectral triples does not always give a well-defined real spectral triple. We refer to Ref. [25] for more details on the graded tensor product of spectral triples, which always yield a real spectral triple). The spectral triple (2.1.22) is called the almost commutative spectral triple.

A geometrical object that one recovers from an almost commutative spectral triple is called almost commutative manifold (AC manifold). The topological structure of the AC manifold is given by the topology on the set of pure states, which is homeomorphic to a product space

$$\mathcal{P}(C^\infty(M) \otimes \mathcal{A}_F) \simeq \mathcal{P}(C^\infty(M)) \times \mathcal{P}(\mathcal{A}_F) \simeq M \times F \ , \tag{2.1.23}$$

or in other words, to a collection of disjoint manifolds. One should note that this is not just a topological space. The space $M \times F$ possesses similar structure to a Riemannian spin manifold, since it has a well-defined KO-dimension (which indicates a generalised notion of spin structure), and the distance function given by the inverse of the Dirac operator (which will be discussed in the next section). The notion of AC manifold allows one to treat the forces of interaction between gauge fields and the force of gravity in an equal footing; all forces arise from the curvature on the AC manifold.

2.2 Infinitesimal and Connes' distance formula

From the manifold reconstruction theorem, it is clear that all geometrical data of a compact Riemannian spin manifold can be encoded into the form of spectral data, therefore, one should anticipate that the geodesic distance can be rewritten in the spectral language. To do so, we begin with the notion of infinitesimal distance. Consider an integrable function f defined on $[0,1]$, and let $\{s_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ be a monotonically decreasing sequence. Let $N \in \mathbb{N}$ such that $s_N \leq 1$, then for $n \geq N$ we have $1 = ms_n + r_n$, for some $m \in \mathbb{N}$, and $0 < r_n \leq s_n$. The integral of the function f is defined by

$$\int_0^1 f(x)d\mu = \lim_{n\to\infty}\left(\sum_{i=0}^{m-1} f(c_i)s_n + f(c_m)r_n\right) \ , \tag{2.2.24}$$

where $c_i \in [x_i, x_i + s_n]$, for $x_0 = 0$ and $i = 0, ..., m-1$, $c_m \in [x_{m-1}, x_{m-1} + r_n]$. Hence, the infinitesimal $d\mu$ can be defined if there exists a monotonically decreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n \to 0$ as $n \to \infty$. This is exactly the property of nonzero eigenvalues of a self-adjoint compact operator, which can be arranged in a decreasing order such that $\lambda_n \to 0$ as $n \to \infty$ [26]. Recall that from the definition of spectral triple, the resolvent of Dirac operator is a compact operator. The compactness of the resolvent operator implies that the inverse $\mathcal{D}^{-1}$ is also compact on the orthogonal complement of its kernel [22], so we define

$$ds := |\mathcal{D}|^{-1}, \tag{2.2.25}$$

where $|\mathcal{D}| = \sqrt{\mathcal{D}^2}$. The existence of infinitesimal suggests that the distance function

$$d(x,y) = \inf_{\gamma} \int_{\gamma} ds \ , \tag{2.2.26}$$

where γ is a curve connecting x and y, can be reformulated in terms of Dirac operator. It was shown by Connes that for a Riemannian manifold (equivalently a canonical spectral triple) the distance function can be rewritten as

$$d(x,y) = \sup\{ \ |f(x) - f(y)| \ ; \ f \in C_0^{\infty}(M), \ \|[\mathcal{D}, f]\| \leq 1\}, \tag{2.2.27}$$

which is known as the Connes' distance formula or the spectral distance formula.

Example 2.2.1. *To show that the formula* (2.2.27) *yields a distance function, let us consider the close interval* $[0,1]$*, which is a compact one-dimensional manifold with the distance function (metric)* $d(x,y) = |x-y|$*. Our goal is to confirm that the rhs of the formula* (2.2.27) *agrees with the distance function. From the condition* $1 \geq \|[\mathcal{D}, f]\|$*, we have*

$$1 \geq \|[\mathcal{D}, f]\| = |\frac{df}{dx}| \ , \tag{2.2.28}$$

which implies that

$$1 \geq \frac{df}{dx} \geq -1 \ . \tag{2.2.29}$$

Integrating inequality (2.2.29) *from* x *to* y*, and then taking supremum one obtains*

$$\sup_f |f(x) - f(y)| \leq |x-y|. \tag{2.2.30}$$

Clearly, the equality is given by $f(x) = x$, so

$$d_{\mathcal{D}}(x, y) = |x - y| = d(x, y). \tag{2.2.31}$$

Notice that using the Gelfand-Naimark theorem, the two points x and y in Eq. (2.2.27) can be replaced by evaluation maps ϵ_x and ϵ_y which are pure states of the C^*-algebra. In this way, one can define a distance function even on a noncommutative spectral triple, namely

$$d_{\mathcal{D}}(\omega, \omega') = \sup\{\ |\omega(a) - \omega'(a)|\ ;\ a \in \mathcal{A},\ \|[\mathcal{D}, a]\| \leq 1\}\ , \tag{2.2.32}$$

where $\omega, \omega' \in \mathcal{P}(\mathcal{A})$. Let us extract some properties of the spectral distance on an almost commutative manifold. Recalls that the set of pure states of almost commutative spectral triple is given by Eq. (2.1.23), the following theorem establishes the Pythagorean relation between the geodesic distance on M and the distance between pure states of the finite-dimensional algebra [27].

Theorem 2.2.2. *Let $\omega_i \in \mathcal{P}(\mathcal{A}_i)$ and $\omega_j \in \mathcal{P}(\mathcal{A}_j)$, and p is a projection such that $p\mathcal{H} = \mathcal{H}_i \oplus \mathcal{H}_j$, where $\mathcal{H}_i, \mathcal{H}_j$ are corresponding Hilbert space of $\mathcal{A}_i, \mathcal{A}_j$ respectively. If $[\mathcal{D}_F, p] = 0$, then the square of spectral distance between $\epsilon_x \otimes \omega_i$ and $\epsilon_y \otimes \omega_j$ is given by [27]*

$$\begin{aligned} d^2(\epsilon_x \otimes \omega_i, \epsilon_y \otimes \omega_j) &= d^2(x, y) + d^2(\omega_i, \omega_j) \\ &= d_M^2(x, y) + d^2(\omega_i, \omega_j)\ , \end{aligned} \tag{2.2.33}$$

where $d_M(x, y)$ is the geodesic distance on M.

Note that for $i = j$, $d(\omega_i, \omega_j) = 0$, therefore, one may think of $\mathcal{P}(\mathcal{A}_i)$ as a point in F, while for $i \neq j$ could give positive number or infinity. Hence, geometrically, F is a finite set of points with finite or infinite separation. If the separation between these points are finite, one can embed the almost commutative manifold $M \times F$ in an $(n+1)$-dimensional Riemannian manifold $M \times \mathbb{R}$. The metric of $M \times \{e_i, e_j\}$ inherited from the ambient manifold is given by

$$g_{ab} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1/d_F(e_i, e_j)^2 \end{pmatrix}, \tag{2.2.34}$$

where greek indices $\mu, \nu \in \{0, 1, 2, 3\}$. In the case of a two-sheets space $M \times \{0, 1\}$ described by the product of the canonical spectral triple with the finite spectral triple

$$A_F = \mathbb{C} \oplus \mathbb{C},\ \mathcal{H}_F = \mathbb{C}^2,\ D_F = \begin{pmatrix} 0 & m \\ m^* & 0 \end{pmatrix}, \tag{2.2.35}$$

where $m \in \mathbb{C}$ is a complex parameter, we have $d_F(0, 1) = 1/|m|$.

2.3 Symmetry, inner fluctuation and gauge fields

It is well-known that the theory of general relativity is invariant under coordinate transformation, or putting in the algebraic language, it is invariant under the action of the diffeomorphism group $\mathrm{Diff}(M)$. A goal of the almost commutative geometry approach is to unify all forces in the spirit of general relativity i.e. all forces arise from curvatures on an almost commutative manifold, hence, one expects the symmetry group to be the "diffeomorphism group" on an almost commutative manifold. To define the analogous notion of the diffeomorphism group, one observes that for $f \in C^\infty(M)$ and $\phi \in \mathrm{Diff}(M)$ the map

$$\alpha_\phi(f) = f \circ \phi^{-1} \tag{2.3.36}$$

is an automorphism on $C^\infty(M)$, therefore, $\mathrm{Diff}(M) \cong \mathrm{Aut}(C^\infty(M))$. Hence, we define the diffeomorphism group on an almost commutative manifold to be

$$\mathrm{Diff}(M \times F) := \mathrm{Aut}(C^\infty(M, \mathcal{A}_F)) \ . \tag{2.3.37}$$

To recover the gauge group of SM, let us consider a particular subgroup of the automorphism group [4]

$$\begin{aligned} G(\mathcal{A}, \mathcal{H}) &:= \{U = uJuJ^* \ ; \ u \in \mathcal{U}(\mathcal{A}), \ \det|_{\mathcal{H}_F} u = 1\} \\ &\cong G_{SM} \times \mu_{12} \ , \end{aligned} \tag{2.3.38}$$

where G_{SM} is the gauge group of SM (one should note that G_{SM} contains elements that act trivially on bosons and fermions). Another important subgroup is the inner automorphism group, denoted by $\mathrm{Inn}(\mathcal{A})$, which is the collection of automorphisms

$$\alpha_u(a) \mapsto uau^* \ , \tag{2.3.39}$$

where $u \in \mathcal{U}(\mathcal{A})$ is an element of the unitary group of the algebra. The group $\mathrm{Inn}(\mathcal{A})$ is isomorphic to the quotient group

$$\mathrm{Inn}(A) \cong \mathcal{U}(\mathcal{A})/\mathcal{U}(Z(\mathcal{A})), \tag{2.3.40}$$

where $Z(\mathcal{A})$ denotes the center of the algebra. Next we will show that gauge fields arise naturally when a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is perturbed by the inner automorphism. The algebra of the perturbed spectral triple is given by $\mathcal{B} := \alpha_u(\mathcal{A}) \cong \mathcal{A}$, but we need the notion of **strong Morita equivalence** to construct the whole spectral triple.

The Morita equivalence between two C^*-algebras $\mathcal{B}$ and $\mathcal{A}$ means that there exists a finitely generated projective right $\mathcal{A}$-module $\mathcal{E}$ (more details on C^*-module can be found in [20]) such that

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \ . \tag{2.3.41}$$

A finitely generated projective module is basically a projection of $\mathcal{A}^N$, for some $N \in \mathbb{N}$, or more precisely, there exists a self-adjoint element $p \in M_N(\mathcal{A})$ such that $p = p^2$ and

$$\mathcal{E} \cong p\mathcal{A}^N. \tag{2.3.42}$$

If any two algebras are Morita equivalent, then one can define the Hilbert space $\mathcal{H}' := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}^0$, where $\mathcal{E}^0 = \{\bar{\xi}; \xi \in \mathcal{E}\}$ is the conjugate module equipped with a left action $a\bar{\xi} = \overline{\xi a^*}$, for $a \in \mathcal{A}$. Suppose $\mathcal{E}$ admits a hermitian connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathcal{D}}$ that satisfies the following conditions [22]

$$\nabla(\xi a) =(\nabla\xi)a + \xi \otimes da \ , \ \forall \xi \in \mathcal{E}, \ a \in \mathcal{A} \ , \tag{2.3.43}$$

$$d\langle \xi, \eta\rangle_{\mathcal{A}} =\langle \xi, \nabla\eta\rangle_{\mathcal{A}} - \langle \nabla\xi, \eta\rangle_{\mathcal{A}} \ , \forall \xi, \eta \in \mathcal{E} \ , \tag{2.3.44}$$

where $\Omega^1_{\mathcal{D}}$ denotes the algebra of one-form, $da := [\mathcal{D}, a]$, and $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$

denotes the hermitian product. Then the Dirac operator can be defined by

$$\mathcal{D}'(\xi \otimes \eta \otimes \bar{\xi}) = \xi \otimes \mathcal{D}\eta \otimes \bar{\xi} + (\nabla\xi)\eta \otimes \bar{\xi} + \xi \otimes \eta\overline{\nabla\xi}. \tag{2.3.45}$$

Since we have $\mathcal{B} \simeq \mathcal{A}$ and $\mathcal{A}$ is Morita equivalence to itself, and we take $\mathcal{E} = \mathcal{A}$. In this case, we denotes the Dirac operator by $\mathcal{D}_A$, which acts on $\psi = 1 \otimes \eta \otimes \bar{1} \in \mathcal{A} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{A}$ as follows

$$\begin{aligned} \mathcal{D}_A(1 \otimes \eta \otimes \bar{1}) &= 1 \otimes \mathcal{D}\eta \otimes \bar{1} + (\nabla 1)\eta \otimes \bar{1} + 1 \otimes \eta\overline{\nabla 1} \\ &= \mathcal{D}\psi + A\psi + \epsilon' JAJ^*\psi \ , \end{aligned} \tag{2.3.46}$$

for $A := \nabla 1 = \sum_j a_j[\mathcal{D}, b_j] \in \Omega^1_{\mathcal{D}}(\mathcal{A})$ is a self-adjoint one-form. The one-form A is called **inner fluctuation**, which will play the role of gauge potential.

To confirm that the inner fluctuations represent gauge fields one needs to observe their behaviour under the unitary transformation. First we introduce the notion of unitary equivalence between spectral triples.

Definition 2.3.1. *Two real spectral triples* $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1, J_1, \gamma_1)$, $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2, J_2, \gamma_2)$ *are unitary equivalent if* $\mathcal{A}_1 = \mathcal{A}_2$, *and there exists a unitary operator* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *such that for* $a_1 \in \mathcal{A}_1$ *and* $a_2 \in \mathcal{A}_2$

$$\begin{aligned} &Ua_1U^* \in \mathcal{A}_2, \quad U\mathcal{D}_1U^* = \mathcal{D}_2 \\ &UJ_1U^* = J_2, \quad U\gamma_1U^* = \gamma_2. \end{aligned} \tag{2.3.47}$$

Given the Dirac operator of the unitary equivalent spectral triple

$$U\mathcal{D}_A U^* = U\mathcal{D}U^* + UAU^* + \epsilon' UJAJ^*U^*, \tag{2.3.48}$$

we will determine the transformation rule of the inner fluctuations. For the first term in Eq. (2.3.48), we have

$$\begin{aligned} U\mathcal{D}U^* &= uJu^*J^*\mathcal{D}JuJ^*u^* \\ &= \epsilon' uJ(\mathcal{D} + u^*[\mathcal{D}, u])J^*u^* \\ &= u\mathcal{D}u^* + \epsilon' Ju[\mathcal{D}, u^*]J^* \\ &= \mathcal{D} + u[\mathcal{D}, u^*] + \epsilon' Ju[\mathcal{D}, u^*]J^* \ . \end{aligned} \tag{2.3.49}$$

Note that we have used the axioms Eq. (2.1.12) for a real spectral triple. Since the right action commutes with any element of $\mathcal{A}$

$$UAU^* = u(Ju^*J^*)AJ^*uJu^* = uA(Ju^*J^*)J^*uJu^* = uAu^*. \tag{2.3.50}$$

Similarly,

$$UJAJ^*U^* = JuAu^*J^*, \tag{2.3.51}$$

therefore,

$$U\mathcal{D}_A U^* = \mathcal{D} + A_u + \epsilon' JA_uJ^*, \tag{2.3.52}$$

where

$$A_u := uAu^* + u[\mathcal{D}, u^*] \ . \tag{2.3.53}$$

The transformation rule Eq. (2.3.53) is exactly the transformation rule of gauge fields, hence, the gauge fields can be derived as inner fluctuations of a Morita equivalent spectral triple.

Let us now compute the inner fluctuations of the spectral triple (2.1.22). Consider $a, b \in C_0^\infty(M, A_F)$, the inner fluctuations read

$$
\begin{aligned}
A + \epsilon' J A J^* &= [a, \mathcal{D} b] + J[a, \mathcal{D} b] J^* \\
&= A_{\not\nabla} + \epsilon' J A_{\not\nabla} J^* + A_{D_F} + \epsilon' J A_{\mathcal{D}_F} J^* \ ,
\end{aligned}
\tag{2.3.54}
$$

where $A_{\not\nabla} := a[-i\gamma^\mu \nabla_\mu \otimes \mathrm{Id}_F, b]$ and $A_{D_F} := a[\gamma^5 \otimes \mathcal{D}_F, b]$. Consider

$$
\begin{aligned}
A_{\not\nabla} + \epsilon' J A_{\not\nabla} J^* &= a[-i\gamma^\mu \nabla_\mu \otimes \mathrm{Id}_F, b] + Ja[-i\gamma^\mu \nabla_\mu \otimes \mathrm{Id}_F, b] J^* \\
&= \ - i\gamma^\mu \otimes a\partial_\mu b + i\gamma^\mu J \otimes a\partial_\mu b J^* \\
&= \gamma^\mu \otimes (-ia\partial_\mu b + iJ_F a \partial_\mu b J_F^*) \\
&=: \gamma^\mu \otimes B_\mu \ ,
\end{aligned}
\tag{2.3.55}
$$

where $B_\mu := -ia\partial_\mu b + iJ_F a\partial_\mu b J_F^*$, and we have used that $J_M \gamma^\mu = -\gamma^\mu J_M$. Next consider

$$
\begin{aligned}
A_{\mathcal{D}_F} + \epsilon' J A_{\mathcal{D}_F} J^* &= a[\gamma^5 \otimes \mathcal{D}_F, b] + Ja[\gamma^5 \otimes \mathcal{D}_F, b] J^* \\
&= \gamma^5 \otimes a[\mathcal{D}_F, b] + \gamma^5 \otimes J_F a[\mathcal{D}_F, b] J_F^* \\
&=: \gamma^5 \otimes \phi \ .
\end{aligned}
\tag{2.3.56}
$$

Hence the fluctuated Dirac operator is given by

$$
\mathcal{D}_A = -i\gamma^\mu \nabla_\mu^E \otimes \mathrm{Id}_F + \gamma^5 \otimes \Phi \ , \tag{2.3.57}
$$

where $\nabla^E_\mu := \nabla^S_\mu \otimes \mathrm{Id}_F + i\mathbb{1}_4 \otimes B_\mu$ is the gauge connection of the principal bundle, and $\Phi := \mathcal{D}_F + \phi$. Using Eq. (2.3.53), one can check that Φ obeys the transformation rule of a scalar field, hence, we shall call it the scalar inner fluctuation.

In addition, the square of the Dirac operator gives a generalised Laplacian

$$\mathcal{D}_A^2 = -g^{\mu\nu}\left(\nabla^E_\mu \nabla^E_\nu - \Gamma^\rho{}_{\mu\nu}\nabla^E_\rho\right) - Q, \tag{2.3.58}$$

the linear map Q is defined by

$$Q = \frac{1}{4}R \otimes \mathrm{Id}_R + \frac{1}{2} i\gamma^\mu\gamma^\nu \otimes F_{\mu\nu} - i\gamma^5\gamma^\mu \otimes D_\mu\Phi - \mathbb{1}_4 \otimes \Phi^2 \ , \tag{2.3.59}$$

where R denotes the Ricci scalar, $D_\mu\Phi := [\nabla^E_\mu, \Phi]$, and

$$F_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu + i[B_\mu, B_\nu] \ . \tag{2.3.60}$$

The generalised Laplacian will play an important role in the next section where we define spectral invariants on an almost commutative manifold.

2.4 Spectral invariants of Dirac operators

Spectral invariants are topological invariants derived from the spectrum of an operator. In this section, we introduce the three important spectral invariants, namely the index of the Dirac operator, the heat kernel, and the spectral zeta function. These invariants can be written in terms of local geometrical invariants e.g. curvatures (for the heat kernel this is true in a certain limit). In addition, we argue that one can define dynamics on spectral geometry by choosing the appropriate spectral invariant.

Fredholm index

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ be an even graded spectral triple. The Hilbert space can be decomposed into the direct sum of positive and negative eigenspaces of γ i.e.

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-, \tag{2.4.61}$$

where

$$\mathcal{H}^+ := \frac{(1+\gamma)}{2}\mathcal{H}, \quad \text{and} \quad \mathcal{H}^- := \frac{(1-\gamma)}{2}\mathcal{H} \;. \tag{2.4.62}$$

Since the grading operator is anticommuting with the Dirac operator, we have

$$\begin{aligned} \mathcal{D}\mathcal{H}^+ &= \mathcal{D}\frac{(1+\gamma)}{2}\mathcal{H} \\ &= \frac{(1-\gamma)}{2}\mathcal{D}\mathcal{H} \subset \mathcal{H}^-. \end{aligned} \tag{2.4.63}$$

Similarly, $\mathcal{D}\mathcal{H}^- \subset \mathcal{H}^+$, hence, the Dirac operator can be represented in the form of a 2×2 matrix

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}, \tag{2.4.64}$$

where $\mathcal{D}_+ : \mathcal{H}^+ \to \mathcal{H}^-$ and $\mathcal{D}_- : \mathcal{H}^- \to \mathcal{H}^+$. Since the Dirac operator is self-adjoint, we have $\mathcal{D}_- = \mathcal{D}_+^*$. The resolvent of the Dirac operator is an inverse of $\mathcal{D}$

up to a compact operator i.e.

$$\begin{aligned}\mathcal{D}(\mathcal{D}-\lambda)^{-1}-\mathrm{Id}_{\mathcal{H}} &= (\mathcal{D}-\lambda+\lambda)(\mathcal{D}-\lambda)^{-1}-\mathrm{Id}_{\mathcal{H}} \\ &= \lambda(\mathcal{D}-\lambda)^{-1} \in \mathcal{K}(\mathcal{H}) \ , \qquad (2.4.65)\end{aligned}$$

similarly $(\mathcal{D}-\lambda)^{-1}\mathcal{D}-\mathrm{Id}_{\mathcal{H}} \in \mathcal{K}(\mathcal{H})$. By Atkinson's theorem [20], the Dirac operator is a linear operator with finite dimensional kernel and Cokernel i.e. $\dim \mathrm{Ker}(\mathcal{D})$, $\dim \mathcal{H}\backslash\mathrm{Ker}(\mathcal{D}) < \infty$. Such an operator is called a Fredholm operator, and we define the Fredholm index for $\mathcal{D}_+$ by

$$\begin{aligned}\mathrm{Index}(\mathcal{D}_+) &:= \dim \mathrm{Ker}(\mathcal{D}_+) - \dim \mathcal{H}\backslash\mathrm{Ker}(\mathcal{D}_+) \\ &= \dim \mathrm{Ker}(\mathcal{D}_+) - \dim \mathrm{Ker}(\mathcal{D}_+^*) \ . \qquad (2.4.66)\end{aligned}$$

Note that, the index of a sefl-adjoint operator is zero. In particular, one cannot define a nontrivial index for a Dirac operator on an odd dimensional manifold, since there is no grading operator on its spinor bundle.

In the following example, we show that the index of the Dirac operator on the Riemannian manifold coincides with the well-known topological invariant, the Euler characteristic number $\chi(M)$.

Example 2.4.1. *de Rham complex*

Let M be a compact Riemannian manifold of dimension $2n$. The exterior derivative is a map $d : \Omega^p(M) \to \Omega^{p+1}(M)$, and the codifferential is $d^* := *d* : \Omega^p(M) \to \Omega^{p-1}(M)$, where $*$ is the Hodge star operator. We define the inner product between *p-form* $\omega, \eta \in \Omega^p(M)$ *as follows*

$$\langle \omega, \eta \rangle := \int_M \omega \wedge *\eta. \qquad (2.4.67)$$

The space of p-forms $\Omega^p(M)$ *with the above inner product yields a Hilbert space (the completion of space of p-forms with respect to the norm defined by the above inner product is a Hilbert space), and the Dirac operator is defined by* $\mathcal{D} := d + d^*$. *One can decompose exterior algebra into a direct sum of even p-forms and odd-p-forms,* $\Omega^{\text{even}}(M) \oplus \Omega^{\text{odd}}(M)$. *Then we have* $(d+d^*)_+ : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$ *and* $(d+d^*)_- : \Omega^{\text{odd}}(M) \to \Omega^{\text{even}}(M)$. *The index of the de Rham complex is*

$$\text{Index}(d+d^*)_+ = \dim\,\text{Ker}(d+d^*)_+ - \dim\,\text{Ker}(d+d^*)_- \ . \qquad (2.4.68)$$

Note that the elements of $\text{Ker}(d+d^*)_+$ *are all even p-forms satisfying*

$$d\omega = 0, \quad \textit{and} \quad d^*\omega = 0 \ . \qquad (2.4.69)$$

Hence, ω *is a harmonic form, and therefore, by Hodge decomposition theorem* $\text{Ker}(d+d^*)_+ \cong \bigoplus_{i=0}^{n} H^{2i}(M, \mathbb{R})$, *the even cohomology class on* M *(the similar argument applies for* $\text{Ker}(d+d^*)_-$*).*

$$\begin{aligned} \dim\,\text{Ker}(d+d^*)_+ - \dim\,\text{Ker}(d+d^*)_- &= \sum_{i=0}^{n} \dim H^{2i}(M, \mathbb{R}) - \sum_{i=0}^{n-1} \dim H^{2i+1}(M, \mathbb{R}) \\ &= \sum_{j=0}^{2n} (-1)^j \dim H^j(M, \mathcal{R}) \\ &= \chi(M). \end{aligned} \qquad (2.4.70)$$

From the above example, one concludes that the index offers a connection between two different worlds of operator theory and geometry. The notion of index is not limited to differential operators, it is well-defined for any $T \in \mathcal{B}(\mathcal{H})$ as long as the dimensions of $\mathrm{Ker}(T)$ and $\mathrm{Ker}(T^*)$(equivalently $\mathrm{Coker}(T)$) are finite, and therefore, it can be generalised to an invariant of a noncommutative spectral triple.

Heat kernel

From the Dirac operator, one defines a Laplacian $\Delta := \mathcal{D}^2 = \mathcal{D}_+^* \mathcal{D}_+ + \mathcal{D}_+ \mathcal{D}_+^*$, which has $\mathrm{Sp}(\Delta) \subset [0, \infty)$. Inspired by the integral kernel of heat equation, the heat kernel is defined by

$$e^{-t\Delta} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (\Delta - \lambda)^{-1} d\lambda \ , \tag{2.4.71}$$

where C is a contour enclosing the spectrum of Δ, and $t > 0$. The heat kernel (2.4.71) is a trace class operator on the Hilbert space [26], hence we define the trace of the heat kernel

$$\mathrm{Tr}_{\mathcal{H}} \left(e^{-t\Delta} \right) .$$

For t sufficiently small, one can think of the heat kernel trace as a way of "counting" the eigenvalues that are smaller than t^{-1}. Suppose λ_n denotes for a nonzero eigenvalue of Δ, then

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}} \left(e^{-t\Delta} \right) &= \dim(\mathrm{Ker}\Delta) + \sum_n^{\infty} e^{-t\lambda_n} \\ &\sim \dim(\mathrm{Ker}\Delta) + \underbrace{1 + 1 + ... + 1}_{N} + C \ , \end{aligned} \tag{2.4.72}$$

for some N such that $t\lambda_N << 1$, and C is a real number. Suppose Δ is a differential operator on a even dimensional Riemannian manifold. The expansion (2.4.72) can be realised as an expansion in Fourier modes

$$\mathrm{Tr}_{\mathcal{H}}\left(e^{-t\Delta}\right) \sim \sum_{k=0}^{\infty} t^{\left(\frac{n-k}{2}\right)} \int_M d^n x \sqrt{|g|} a_k(x,\Delta) = \sum_{k=0}^{\infty} t^{\left(\frac{n-k}{2}\right)} a_k(\Delta) \ , \qquad (2.4.73)$$

where each even Fourier coefficient $a_{2m}(x,\Delta)$ is a linear combinations of invariant polynomials of curvature and their derivatives, while odd one vanishes i.e. $a_{2m+1}(x,\Delta) = 0$. Although the heat trace expansion is an approximation at $t \to 0$, it allows one to write an alternative formula for the index of the Dirac operator.

Proposition 2.4.2. *Let $\mathcal{D}$ be the Dirac operator on an even spectral triple. The index is given by*

$$\mathrm{Index}(\mathcal{D}_+) = \mathrm{Tr}_{\mathcal{H}}\left(\gamma e^{-t\Delta}\right) = \mathrm{Tr}_{\mathcal{H}}\left(e^{-t\mathcal{D}_+^*\mathcal{D}_+}\right) - \mathrm{Tr}_{\mathcal{H}}\left(e^{-t\mathcal{D}_+\mathcal{D}^*{}_+}\right) \qquad (2.4.74)$$

Proof. Since $\mathcal{D}_+^*\mathcal{D}_+$ and $\mathcal{D}_+\mathcal{D}_+^*$ share the same nonzero eigenvalues, we have

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}}\left(e^{-t\mathcal{D}_+^*\mathcal{D}_+}\right) - \mathrm{Tr}_{\mathcal{H}}\left(e^{-t\mathcal{D}_+\mathcal{D}_+^*}\right) &= \mathrm{Tr}_{\mathcal{H}}P_+ - \mathrm{Tr}_{\mathcal{H}}P_- \\ &= \dim\,\mathrm{Ker}(\mathcal{D}_+^*\mathcal{D}_+) - \dim\,\mathrm{Ker}(\mathcal{D}_+\mathcal{D}_+^*)\end{aligned}$$

where P_+ and P_- are projections onto kernels of $\mathcal{D}_+^*\mathcal{D}_+$ and $\mathcal{D}_+\mathcal{D}_+^*$ respectively.

Since $\Delta = \mathcal{D}^2$, it is clear that $\mathrm{Ker}(\Delta) = \mathrm{Ker}(\mathcal{D})$, hence,

$$\begin{aligned}\mathrm{Ker}(\mathcal{D}_+^*\mathcal{D}_+) \oplus \mathrm{Ker}(\mathcal{D}_+\mathcal{D}_+^*) &= \mathrm{Ker}(\Delta) \\ &= \mathrm{Ker}(\mathcal{D}) \\ &= \mathrm{Ker}(\mathcal{D}_+) \oplus \mathrm{Ker}(\mathcal{D}_+^*)\ .\end{aligned}$$

The domains of $\mathcal{D}_+^*\mathcal{D}_+$ and $\mathcal{D}_+$ both lie within $\mathcal{H}^+$, while the domains of $\mathcal{D}_+\mathcal{D}_+^*$ and $\mathcal{D}_+^*$ lie within $\mathcal{H}^-$, therefore,

$$\mathrm{Ker}(\mathcal{D}_+^*\mathcal{D}_+) = \mathrm{Ker}(\mathcal{D}_+), \quad \text{and} \quad \mathrm{Ker}(\mathcal{D}_+\mathcal{D}_+^*) = \mathrm{Ker}(\mathcal{D}_+^*)\ .$$

Hence

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}}\left(e^{-t\mathcal{D}_+^*\mathcal{D}_+}\right) - \mathrm{Tr}_{\mathcal{H}}\left(e^{-t\mathcal{D}_+\mathcal{D}_+^*}\right) &= \dim\,\mathrm{Ker}(\mathcal{D}_+) - \dim\,\mathrm{Ker}(\mathcal{D}_+^*) \\ &= \mathrm{Index}(\mathcal{D}_+)\ .\end{aligned}$$

$\square$

In the case that the Dirac operator is a differential operator, using the proposition 2.4.2 and the heat trace expansion, we have

$$\mathrm{Index}(\mathcal{D}_+) = \sum_{k=0}^{\infty} t^{\left(\frac{n-k}{2}\right)} \int_M d^n x \sqrt{|g|}\left(a_k(x, \mathcal{D}_+^*\mathcal{D}_+) - a_k(x, \mathcal{D}_+\mathcal{D}_+^*)\right). \qquad (2.4.75)$$

Since the lhs of (2.4.75) is independent of t, one concludes that the index is given by

$$\mathrm{Index}(\mathcal{D}_+) = \int_M d^n x \sqrt{|g|}\left(a_n(x, \mathcal{D}_+^*\mathcal{D}_+) - a_n(x, \mathcal{D}_+\mathcal{D}_+^*)\right). \qquad (2.4.76)$$

Hence we prove the well-known local index formula using the heat kernel expansion.

We mentioned in the previous section that the square of Dirac operator Eq. (2.3.58) gives a second order differential operator, therefore, the heat trace expansion can be applied to an almost commutative manifold. In such a case the first three Seeley-DeWitt coefficients are given by

$$\begin{aligned}
a_0(\Delta) &= (4\pi)^{-2}\int d^4x\sqrt{|g|}\mathrm{Tr}_{\mathcal{H}}\left(Id\right)\\
a_2(\Delta) &= (4\pi)^{-2}\int d^4x\sqrt{|g|}\mathrm{Tr}_{\mathcal{H}}\left(-\frac{R}{6}+Q\right)\\
a_4(\Delta) &= (4\pi)^{-2}\int d^4x\sqrt{|g|}\mathrm{Tr}_{\mathcal{H}}\big(12\triangle R+5R^2-2R_{\mu\nu}R^{\mu\nu}+2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\\
&\qquad -60RQ+180Q^2-60\triangle Q+30\Omega^E_{\mu\nu}(\Omega^E)^{\mu\nu}\big)\,, \qquad (2.4.77)
\end{aligned}$$

where $R_{\mu\nu\rho\sigma}, R_{\mu\nu}$ and R stand for the Riemannian curvature, the Ricci tensor and the Ricci scalar respectively. The linear map Q is previously defined by Eq. (2.3.59), and the curvature two-form is given by the commutator of the principal connections

$$\begin{aligned}
\Omega^E_{\mu\nu} &= [\nabla^E_\mu, \nabla^E_\nu]\\
&= [\nabla^S_\mu, \nabla^S_\nu]\otimes \mathrm{Id}_F + i\mathbb{1}_4\otimes\partial_\mu B_\nu\\
&\quad - i\mathbb{1}_4\otimes\partial_\nu B_\mu - \mathbb{1}_4\otimes[B_\mu, B_\nu]\\
&= \frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma\otimes\mathrm{Id} + i\mathbb{1}_4\otimes F_{\mu\nu}\ , \qquad (2.4.78)
\end{aligned}$$

where $F_{\mu\nu} := \partial_\mu B_\mu - \partial_\nu B_\nu + i[B_\mu, B_\nu]$. Notice that, $a_2(\Delta)$ contains the Einstein-Hilbert action, and $a_4(\Delta)$ contains the kinetic terms of gauge fields. Such terms inspire the definition of the bosonic spectral action, which gives dynamics to bosonic fields on an almost commutative space (in the next Chapter we will see how the spectral action gives the action of SM coupled to gravity). The bosonic spectral action is

$$S_b = \mathrm{Tr}_{\mathcal{H}} f(\mathcal{D}^2/\Lambda^2) = \int_0^\infty dt f(t) \mathrm{Tr}_{\mathcal{H}} e^{-t\mathcal{D}^2/\Lambda^2} \ , \tag{2.4.79}$$

where f is a positive cutoff function and Λ a cutoff scale. The fermionic part of the action, which does not involve the heat kernel, is defined by

$$S_f = \langle J\psi, \mathcal{D}\psi\rangle \ , \tag{2.4.80}$$

for ψ is a grassmann variable in $+1$-eigenspace of the grading γ.

Zeta function

Let P be an elliptic operator with $\mathrm{Sp}(P) \subset (0, \infty)$. The spectral zeta function $\zeta(s, P)$ is defined as

$$\zeta(s, P) := \mathrm{Tr} P^{-s} = \frac{1}{2\pi i}\int_C \lambda^{-s} \mathrm{Tr}_{\mathcal{H}}(P-\lambda)^{-1} d\lambda \ , \tag{2.4.81}$$

where C is a contour enclosing $\mathrm{Sp}(P)$. The expression is the meromorphic extension (the extension is well-defined and analytic except at countably many poles) of the zeta function

$$\zeta(s, P) = \sum_{n=1}^{\infty} \lambda_n^{-s} \ , \tag{2.4.82}$$

which converges for sufficiently large $\mathrm{Re}(s)$. Since a Laplacian can have zero eigenvalues, to define the zeta function for a Laplacian, one needs to modify the operator in the following way

$$\Delta_\varepsilon := \Delta + \varepsilon P_{\text{Ker}} \ , \tag{2.4.83}$$

for $\varepsilon > 0$ and P_{Ker} is the projection onto the kernel of Δ. Using the definition of the gamma function

$$\lambda^{-s}\Gamma(s) = \int_0^\infty t^{s-1} e^{-\lambda t} dt \ , \tag{2.4.84}$$

one can rewrite the zeta function in terms of the heat kernel trace

$$\zeta(s, \Delta_\varepsilon) = \Gamma(s)^{-1} \int_0^\infty t^{s-1} \text{Tr}_{\mathcal{H}} \left(e^{-t\Delta_\varepsilon}\right) dt \ . \tag{2.4.85}$$

The dependency on ε can be separated from the zeta function as follows

$$\begin{aligned}
\zeta(s, \Delta_\varepsilon) &= \Gamma(s)^{-1} \int_0^\infty t^{s-1} \text{Tr}_{\text{Ker}(\Delta)} \left(e^{-\varepsilon t P_{\text{Ker}}}\right) dt + \Gamma(s)^{-1} \int_0^\infty t^{s-1} \text{Tr}_{\lambda > 0} \left(e^{-t\Delta}\right) dt \\
&= \Gamma(s)^{-1} \int_0^\infty t^{s-1} e^{-\varepsilon t} \text{Tr}_{\mathcal{H}} \left(P_{\text{Ker}}\right) dt + \Gamma(s)^{-1} \int_0^\infty t^{s-1} \text{Tr}_{\lambda > 0} \left(e^{-t\Delta}\right) dt \\
&= \varepsilon^{-s} \text{dim Ker}(\Delta) + \Gamma(s)^{-1} \int_0^\infty t^{s-1} \text{Tr}_{\lambda > 0} \left(e^{-t\Delta}\right) dt \ .
\end{aligned} \tag{2.4.86}$$

The notation $\text{Tr}_{\lambda>0}$ denotes the trace on the eigenbasis e_n with $\lambda_n \neq 0$, while $\text{Tr}_{\text{Ker}(\Delta)}$ denotes the trace on the kernel of Δ. The immediate consequence of Eq. (2.4.86) is that the ζ function is independent of ε at $s = 0$. The regularity of the zeta function at $s = 0$ is discussed in [28] in the context of almost commutative manifolds, and in [29] in a more general noncommutative setup. In particular, we have

$$\zeta(s, \gamma\Delta_\varepsilon) = \varepsilon^{-s} \left(\text{dimKer}(\mathcal{D}_+) - \text{dimKer}(\mathcal{D}_+^*)\right) = \varepsilon^{-s} \text{Index}(\mathcal{D}_+) \ , \tag{2.4.87}$$

therefore

$$\mathrm{Index}(\mathcal{D}_+) = \zeta(0, \gamma\Delta_\varepsilon) \ . \tag{2.4.88}$$

In the following we will write $\zeta(0, \Delta)$ instead of $\zeta(0, \Delta_\varepsilon)$, since the function does not depend on the choice of ε.

Since the zeta function can be expressed in terms of heat kernel trace, its local formula can be derived from the heat kernel expansion. The proof of this statement can be found in Ref. [30, 26]. The local formula of the zeta function on an n-dimensional manifold is given by

$$\zeta(0, \Delta) = \int_M d^n x \sqrt{|g|} a_n(x, \Delta) \ . \tag{2.4.89}$$

Unlike the heat kernel whose local formula (heat trace expansion) only valid for small t, Eq. (2.4.89) is always valid on a closed manifold.

Chapter 3

Spectral action with zeta function regularisation

The spectral action Eq. (2.4.79) of an almost commutative manifold is defined based on the heat kernel trace of a Laplacian. This definition allows one to write down the full action in a compact form, but it is practically impossible to perform any calculation. The actual calculation which leads to phenomenological predictions is only possible through the heat trace expansion. However, the need of the expansion raises the question of the physical meaning of the cut-off scale Λ, and other issues such as convergence, locality, and renormalisability. The main aim of this Chapter is to introduce a new action functional on an almost commutative space, which addresses some of these open issues.

The contents of this Chapter are organised as follows. In the first section, we describe how one can obtain the action of the Standard Model from the spectral action of an almost commutative space, and the phenomenological predictions from the spectral action. In section 3.2 we list some open issues in the cut-off spectral action approach, and then in section 3.3 we address some of these issues by introducing a new action defined by the spectral zeta function. In the last section we study the ultraviolet behaviour of the gravitational action derived from the zeta spectral action by calculating its spectral dimension.

3.1 Asymptotic expansion of a spectral action

In the previous Chapter we introduced the formal expression of the spectral action, which is practically impossible to extract phenomenological information. To obtain an action that will allow us to extract physics, one needs an approximation scheme that preserves the relevant information of SM. First, note that the highest energy we can observe in the laboratory is in the TeV scale. One can always pick an energy scale Λ that is much higher than the TeV scale and assume that the physics at an energy higher than Λ does not interfere with physics at TeV scale. Since Λ^{-2} is very small the heat kernel trace can be expanded, and the action (2.4.79) can be approximated by

$$\mathrm{Tr}_{\mathcal{H}} f(\mathcal{D}_A^2/\Lambda^2) \sim 2f_4 a_0(\mathcal{D}_A)\Lambda^4 + 2f_2 a_2(\mathcal{D}_A)\Lambda^2 + f(0)a_4(\mathcal{D}_A) + O(\Lambda^{-2}) \ , \quad (3.1.1)$$

where $\mathcal{D}_A$ is the inner fluctuated Dirac operator defined previously, and

$$f_{4-k} := \int_0^\infty x^{4-k-1} f(x)dx \quad , \quad 0 \leq k < 4 \ . \qquad (3.1.2)$$

From Eq. (2.4.72), since Λ is much higher than TeV scale, the dominated term in the expansion corresponds to the eigenvalues that are relevant to the physics at TeV scale. Substituting the coefficients from Eq. (2.4.77) in the above expansion, the spectral action reads [4, 31]

$$S_b \sim \int \sqrt{|g|} \left[\frac{48 f_4}{\pi^2}\Lambda^4 - \frac{c f_2}{\pi^2}\Lambda^2 + \frac{d f(0)}{4\pi^2} + \left(\frac{4 f_2}{\pi^2}\Lambda^2 - \frac{c f(0)}{24\pi^2} \right) R - \frac{3 f(0)}{10\pi^2} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \right.$$
$$+ \frac{1}{4} Y_{\mu\nu} Y^{\mu\nu} + \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu,a} + \frac{1}{4} G^i_{\mu\nu} G^{\mu\nu,i} + \frac{1}{2} |D_\mu H|^2 - \frac{1}{12} R H^2$$
$$\left. - \frac{2 a f_2 \Lambda^2 - e f(0)}{a f(0)} H^2 + \frac{b \pi^2}{2 a^2 f(0)} H^4 \right] d^4 x + \mathcal{O}(\Lambda^{-2}) \ , \qquad (3.1.3)$$

where

$$D_\mu H := \partial_\mu H + \frac{1}{2} i g_2 W^a_\mu \sigma^a H - \frac{1}{2} i g_1 A_\mu H \ , \qquad (3.1.4)$$

and the constants a, b, c, d and e are derived from Yukawa and Majorana mass matrices. The field strength tensors $Y_{\mu\nu}, W_{\mu\nu}$ and $G_{\mu\nu}$ are derived from the gauge fields A_μ, W_μ and G_μ which belong to the Lie algebra of the symmetry groups $\mathrm{U}_Y(1)$, SU(2) and SU(3) respectively. The field H, representing the Higgs field, is derived from the scalar inner fluctuation as follows

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}_F}(D_\mu \Phi D^\mu \Phi) &=: \frac{4\pi^2}{f(0)} |D_\mu H|^2 \\ \mathrm{Tr}_{\mathcal{H}_F}(\Phi^2) &=: \frac{4\pi^2}{f(0)} |H|^2 + 2c \\ \mathrm{Tr}_{\mathcal{H}_F}(\Phi^4) &=: \frac{4 b \pi^4}{f(0)^2 a^2} |H|^4 + \frac{8 e \pi^2}{f(0) a} |H|^2 + 2d \ . \end{aligned}$$

The requirement that the kinetic terms of gauge fields are normalised, results in the following relation between gauge couplings

$$\frac{5}{3} g_1^2 = g_2^2 = g_3^2 \ , \qquad (3.1.5)$$

for g_1, g_2 and g_3 are the couplings associated with the three gauge fields A_μ, W_μ and

G_μ respectively. Hence, the action should be naturally defined at grand unification scale (GUT scale). Although it is well-known that the renormalisation group flow of the Standard Model does not allow the equality (3.1.5) at any energy scale, the Eq. (3.1.5) approximately holds around the energy scale $\Lambda \sim (10^{14} - 10^{17})$ GeV. We shall proceed by assuming that the spectral action is defined at $\Lambda = 10^{17}$. Since the spectral action is defined at such a high energy scale, the predictions are obtained by solving the renormalisation group equation of masses and coupling at the energy scale where the experiment is conducted (note that the calculation is performed on flat spacetime). It is shown [4] that these predictions are in agreement with the experimental values, except the mass of the Higgs boson which is predicted to be around 170 GeV. The correct Higgs mass was achieved by proposing the existence of a new scalar field [5], which could be obtained by modifying the algebra, or relaxing the first order axiom [6, 7, 8, 9].

Despite the considerable achievements of almost commutative geometry (ACG) approach to the Standard Model, the spectral action Eq. (3.1.3) still leaves open some important issues.

3.2 Open issues in the cut-off spectral action approach

The following are the list of some open issues in the cut-off spectral action approach.

I. Locality in the high momentum regime: The spectral action is highly dependent on the cut-off scale Λ. Although in the low momentum regime the expansion (3.1.1) recovers the Standard Model action, the high momentum regime does not contain positive powers of the field derivatives [32, 33], exhibiting the structure

$$S_\Lambda \sim \int d^4x \left(\alpha_1 \, \Lambda^2 \, h_{\mu\nu} \, h^{\mu\nu} + \alpha_2 \, \phi \, \frac{\Lambda^4}{\partial^2} \, \phi + \alpha_3 \, A_\mu \, \frac{\Lambda^4}{\partial^2} \, A^\mu \right) \ , \tag{3.2.6}$$

where ϕ and A_μ are bosons of spin 0 and 1 respectively; $\alpha_{1,2,3}$ are constants depending on the particular realisation of the model. The transverse and traceless fluctuations $h_{\mu\nu}$ of the metric tensor $g_{\mu\nu}$ are defined as follows

$$g_{\mu\nu} = \delta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{\mathrm{Pl}}}, \tag{3.2.7}$$

where M_{Pl} is the Planck mass, i.e. they have canonical dimension of energy. This opens the question of the meaning of the cut-off scale Λ, and what happens beyond it.

II. Renormalisability: It is shown in Ref. [34] that on the flat space the action Eq. (3.1.3) is renormalisable, given that the $O(\Lambda^{-2})$ is suppressed by the cutoff. However, the full spectral action is certainly not renormalisable, as it stands, since at high momenta the bosonic propagators do not decrease. For instance, in contrast to conventional QED, the diagram presented in Fig. 3.1 is divergent, therefore one has to add four fermonic interactions $\left(\bar{\psi}\psi\right)^2$ in order to subtract the infinity. Theories with four fermonic interactions are well known to be nonrenormalisable.

III. Convergence and predictive power: The spectral action (3.1.3) is calculated via the asymptotic heat kernel expansion. This expansion can be divergent and generally speaking does not coincide with the spectral action [35]. From a predictive point of view there is also an unpleasant dependence on the particular shape

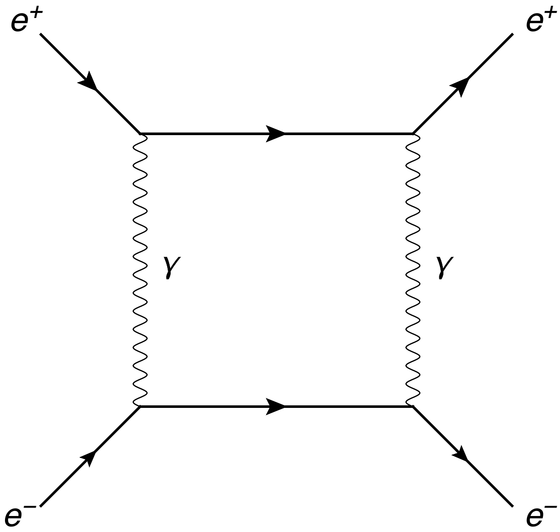

Figure 3.1: *We present an ultraviolet divergent diagram leading to the introduction in the theory of four fermionic vertex, i.e. making it nonrenormalisable. Wavy lines present photon propagators, arrowed lines correspond to electrons and positrons.*

of the cutoff function, whose momenta f_n appear in the asymptotic expansion in inverse powers of Λ [36]. Strictly speaking this dependence introduces infinite number of extra parameters.

IV. Naturalness: Another issue is that the magnitude of the *dimensionful* parameters appearing in the model, the cosmological constant, the Higgs vacuum expectation value and the gravitational coupling have to be put in (3.1.3) by hand. We emphasise that independently of the choice of the almost commutative manifold, the physical values of these quantities necessitate an experimental input which goes beyond the data encoded by the spectral triple. All these quantities have to be substituted by a subtraction point which fixes their value by hand to fit the experimental data. This drawback is closely related to the naturalness problem.

V. Lorentzian signature: The spectral action is derived in an almost commutative space $M \times F$, where M is a four-dimensional Riemannian manifold, hence, the tensors appearing in the asymptotic expansion are $SO(4)$ invariants; the asymptotic action describes nonrelativistic dynamics. To obtain relativistic theory, one performs the Wick rotation that turns $SO(4)$ invariant tensors into $SO(1,3)$ invariant tensors. Although a_n can be written in terms of $SO(1,3)$ tensors, the heat kernel trace itself is ill-defined on a pseudo-Riemannian manifold (this is related to the fact that the heat kernel trace and the expansion do not generally coincide). Hence, strictly speaking, one cannot directly obtain an action of the Standard Model from the spectral action of an almost commutative geometry, but rather the action "inspired by the asymptotic expansion of the spectral action."

The problems *I-III* will be addressed in this Chapter, while in order to deal with the problems *IV*-*V*, we believe that more experimental input and a better understanding of Lorentzian spectral triple (a practical definition is given in Chapter 5) are required.

3.3 Zeta function regularisation

Going back to the origins of the bosonic spectral action, one notes that this is a regularised version of the number of eigenvalues of the square of the Dirac operator. The number of eigenvalues of an unbounded operator is of course infinite and one has to (classically) regularise this sum, which would be otherwise $1+1+1\ldots$. The spectral action does it with the introduction of the cutoff scale Λ.

To cure some of the drawbacks of the conventional bosonic spectral action outlined above, we need an action functional with the following basic properties:

- represent the regularised sum of the number of eigenvalues of $\mathcal{D}^2$
- can always be written as integral of local invariants
- does not depend on any parameter

The spectral zeta function $\zeta(0, \mathcal{D}^2)$ has all of these properties, hence we propose the following definition.

Definition 3.3.1. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \gamma)$ be a real even spectral triple. The zeta spectral action is given by*

$$S_\zeta := \zeta(0, \mathcal{D}^2) \ . \tag{3.3.8}$$

From Eq. (2.4.89), we write the zeta spectral action in terms of local curvature invariants

$$S_\zeta = \int d^4x \sqrt{g}\, a_4(\mathcal{D}^2, x) := \int d^4x \sqrt{g}\, \mathscr{L} \ . \tag{3.3.9}$$

Here we use the zeta function to define the classical action, while in a slightly different context the zeta function regularisation is also commonly used to regularise functional determinants appearing upon quantisation [37]. The spectral action (3.3.9) is nothing but the conformal anomaly in a theory of quantised fermions [38] where the bosonic fields are a classical background, the relation between the cutoff spectral action and the anomaly can be found in Refs. [39, 40, 41, 42]. The Lagrangian density obtained from the ζ spectral action has the form:

$$\begin{aligned}
\mathscr{L}(x) \ = \ & \alpha_1 d + \alpha_2 cR + \alpha_3 cH^2 \\
& + \alpha_4 Y_{\mu\nu} Y^{\mu\nu} + \alpha_5 W^{\alpha}_{\mu\nu} W^{\mu\nu\,\alpha} + \alpha_6 G^{a}_{\mu\nu} G^{\mu\nu\,a} \\
& + \alpha_7\, H \left(-\nabla^2 - \frac{R}{6} \right) H + \alpha_8 H^4 + \alpha_9 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \alpha_{10} R^* R^* \ ,
\end{aligned} \tag{3.3.10}$$

where $\alpha_1, .., \alpha_{10}$ are dimensionless constants determined by the Dirac operator (whose particular form is not relevant here); R^*R^* is the Gauss-Bonnet density and C is the Weyl tensor. The number d and c, which are the same as in the spectral action (3.1.3), are defined by

$$d = \mathrm{tr}\left((Y_R^* Y_R)^2\right), \quad \text{and} \quad c = \mathrm{tr}\left(Y_R^* Y_R\right) , \tag{3.3.11}$$

where Y_R is the Majorana mass matrix. Clearly d and c have mass dimension 4 and 2 respectively.

Let us comment here on the fundamental role played by the dimensionful constant M appearing in the position corresponding to the Majorana mass in the Dirac operator. The bare values of the cosmological constant, Higgs mass parameter and the gravitational constant must be renormalised, so at a first look these terms do not carry any predictive power. This is not correct: these terms define the structure of the counter terms needed to eliminate divergences upon quantisation when one uses dimensional regularisation. Indeed, if one has $Y_R = 0$, since there are no dimensionful constants in the bare Lagrangian anymore, divergences proportional to 1, R and H^2 would not appear, and there would be no necessity to introduce the corresponding counter terms. Correspondingly, the cosmological constant, Higgs mass parameter and the gravitational constant would never come out from renormalisation. In the context of the spectral action the Majorana mass term already plays a fundamental role for the phenomenological viability of the model; in the present context its role is even enhanced.

The bosonic spectral action S_ζ contains only terms needed for the Standard Model and Einstein gravity and *nothing else* (e.g. higher dimensional operators) therefore it is *local*, *renormalisable* and *unitary*[1]. This means that one can use renormalisation and safely compute an arbitrary loop order corrections. Another strong advantage of the definition (3.3.9) is the fact that the Lagrangian (3.3.10) is an exact result, therefore there is no need to consider asymptotic expansions and their convergence. From the noncommutative geometry point of view the added advantage is that S_ζ is purely spectral, i.e. it is defined just via the Dirac operator and there is no dependence on a cutoff function.

Substituting the Weyl square and Gauss-Bonnet density expressions via R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ we can rewrite our Lagrangian as a linear combination

$$\mathscr{L}(x) = \sum_{j=1}^{12} \eta_j O_j, \tag{3.3.12}$$

where

$$\begin{aligned} &O_1 = 1, \quad O_2 = R, \quad O_3 = H^2, \quad O_4 = Y_{\mu\nu}Y^{\mu\nu}, \quad O_5 = W^{\alpha}_{\mu\nu}W^{\mu\nu\,\alpha}, \\ &O_6 = G^{a}_{\mu\nu}G^{\mu\nu\,a}, \quad O_7 = H\nabla^2 H, \quad O_8 = H^2 R, \quad O_9 = H^4, \quad O_{10} = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \\ &O_{11} = R_{\mu\nu}R^{\nu}, \quad O_{12} = R^2 \ . \end{aligned} \tag{3.3.13}$$

The Lagrangian given by Eq.(3.3.13) is the most general *renormalisable* Lagrangian for QFT in curved spacetime[2], and correspondingly the complete spectral action

$$S = \langle J\psi, \mathcal{D}_A \psi\rangle + S_\zeta \ , \tag{3.3.14}$$

[1]For the issue of renormalisability and unitarity gravity is still a classical background. We comment more in the outlook.

[2]For renormalisation of QFT in 4-dimensional curved spacetime and corresponding counter terms see e.g. Ref. [43].

is a renormalisable theory describing the Standard Model. Upon quantisation all twelve composite operators O_j in Eq. (3.3.13) must be renormalised and after proper introduction of the renormalisation matrix and counter terms the coefficients η_j by the end of the day must be replaced by renormalised physical parameters $\eta_j^{\rm phys}$. Quantum field theory never predicts the physical values of the coefficients $\eta_j^{\rm phys}$ and they must be fixed at some energy scale by normalisation conditions. Usually such normalisation is done using the values obtained from experiment at low energy[3]. For the spectral action however it is natural to fix the scale at the unification point, and this fixes the relations with all other coefficients, which likewise are normalised at the unification point, *with their value given by the spectral action.* We emphasise that this normalisation procedure is not a consequence of the spectral geometry framework, but is a natural prescription. This prescription gives predictive power, and we will use it considering the scale at which the ζ spectral action is written to be $\sim(10^{14}-10^{17})$ GeV. In analogy with the conventional bosonic spectral action discussed in Sec. 2.4, we will still call this scale Λ. In conclusion, the bosonic spectral action is written as an action valid at a particular scale, whilst the action is itself independent of this scale.

Remark. *It is useful to compare our approach with the one of Ref.* [44] *where the spectral action was defined by the ansatz of the (generally divergent) asymptotic expansion*

$$\sum_{n=0}^{N} f_{2n}\Lambda^{4-2n}a_{2n}[\mathcal{D}], \tag{3.3.15}$$

where f_n are arbitrary and $N\geq 2$. This makes the theory local and super renormalisable, with Λ a cutoff, not a physical scale.

[3]Here "low" may mean TeV scale, which is still much lower than the unification scale.

The higher terms are a particular kind of higher derivative regularisation [45], *in particular when $N = 3$ we have the following action for the gauge field*

$$f_4 F^a_{\mu\nu} F^{\mu\nu\ a} + \frac{f_6}{\Lambda^2} F^a_{\mu\nu} (-\partial)^2 F^{\mu\nu\ a}, \tag{3.3.16}$$

which improves the ultraviolet behaviour of the propagator

$$\frac{1}{p^2} \to \frac{1}{p^2 + \frac{f_6}{f_4 \Lambda^2} p^4}. \tag{3.3.17}$$

At finite values of Λ such theories are known to be super renormalisable (but with ghosts) and in the limit $\Lambda \to \infty$ one recovers the original renormalisable (without ghosts) theory. Since there are still divergent one loop fermionic diagrams one would then have to regularise the theory with dimensional regularisation, thereby creating an artificial hybrid of higher-derivative and dimensional regularisations [46]. *For $N = 2$ in flat spacetime the action is renormalisable and unitary. However, the coefficients a_0 and a_2 that are supposed to introduce the cosmological constant, Higgs vacuum expectation value and Einstein-Hilbert action term do not have by themselves predictive power, since all these parameters have to be normalised using experimental values. If, keeping $N = 2$, one removes them by hand, the definition* (3.3.15) *will lead to our definition* (3.3.8).

There remains to discuss in more detail the issue of lower (less than four) dimension operators, especially in relation to the different scales which have to be introduced, and the corresponding hierarchy. The predictive power of the lower heat kernel coefficient a_0 and a_2 is substantially different from the one of a_4, which contains all dimension four operators. In particular, the coefficients in front of lower dimensional operators are not obtained from spectral data, therefore their normalisation must be imposed by hand using experimental data. This is closely related with the problem of the hierarchy of their numerical value (often called the naturalness problem).

The minimal Standard Model Lagrangian in the presence of gravity can be written as[4]

$$S_{\rm SM} = \int d^4x \sqrt{g} \left(\bar{\Lambda} + \frac{M_{\rm Pl}^2}{16\pi} R + \lambda (H^2 - v^2)^2 + ... \right) \ , \tag{3.3.18}$$

and involves three dimensionful parameters, which we express as energies:

- *the cosmological constant:* $\bar{\Lambda} \sim (10^{-12}{\rm GeV})^4$;
- *the Higgs vacuum expectation value:* $v \sim 10^3$ GeV ;
- *the Plank scale:* $M_{\rm Pl} \sim 10^{19}$ GeV .

All three parameters are *empirical* quantities: the first one describes the rate of expansion of the Universe and its value is deduced from *observational* cosmological data [47], the second one is responsible for the *experimentally* obtained masses of quarks and electroweak bosons [48], while the last one has to do with the Newtonian attraction, namely the gravitational constant, and its value is also known from from *experiments* [49]. The vastly different values of these three scales lead to the hierarchy problem in the minimal Standard Model. Clearly any input on the origin of these dimensionful constants and their vastly different values would be an important achievement. Several attempts have been made; some rely on the

[4] Here λ is a quartic *dimensionless* coupling and "..." stand for other terms of the SM Lagrangian, that contain only dimensionless Yukawa and gauge couplings.

geometry of spacetime, like the addition of extra spatial dimensions leading to a large value of $M_{\rm Pl}$ as an effective value from a scale of a few TeV.

The spectral approach is very successful giving restrictions on *dimensionless* parameters like Higgs quartic coupling, gauge couplings and Yukawa couplings, etc. For the *zeta* spectral action in its present formulation the issue is the value of the dimensionful constants in the lower dimensional terms in the action. We have already seen that the presence of the Majorana mass term in the Dirac operator introduces the correct lower dimensional operators, however the corresponding coefficients are physically inappropriate. Therefore these three numbers can not be taken from the spectral action, and one has to normalise the lower dimensional operators by hand, thereby leaving the naturalness problem unsolved.

The above discussion leads us to propose the following formulation of the normalisation procedure.

By definition:

- all dimensionless constants, i.e. all except the three involved in the naturalness problem are normalised to their *spectral* values.
- three parameters, the cosmological constant, the Higgs mass parameter, and the Planck mass creating the naturalness problem are normalised using experimental input.

We emphasise, that although we weaken the conventional normalisation prescription in the "hierarchy problematic sector", we do not lose predictive power with respect to the original approach, where the cosmological constant, the Higgs mass parameter and the gravitational constant were as well not predicted from the spectral data, while in addition in the cutoff spectral action approach one had the additional freedom to adjust parameters using the ambiguity of choice of the cutoff function.

3.4 Gravitational spectral dimension

The spectral dimension is the dimension of spacetime as observed by a diffusing test particle. The diffusion equation is controlled by a Laplace-Beltrami operator, which is an elliptic operator. This notion can be extended to a more general elliptic operator P such that its spectrum is bounded from below

Definition 3.4.1. *Let a vector space V be a fibre of a vector bundle with a base manifold M. Let $K(x,x',T)$, for $T \in \mathbb{R}$, be the heat kernel of an elliptic operator $P : C^\infty(M,V) \to C^\infty(M,V)$ such that $\mathrm{Sp}(P) \subset [c,\infty)$, for some $c \in \mathbb{R}$. The running spectral dimension* [50] *is a complex value function defined by*

$$\widetilde{D_\mathrm{s}}(T) := -2\frac{\partial \log \mathrm{Tr}_V K(x,x,T)}{\partial \log T} \ , \tag{3.4.1}$$

and the spectral dimension is given by the limit

$$D_\mathrm{s} := \lim_{T\to 0} \widetilde{D_\mathrm{s}}(T) \ . \tag{3.4.2}$$

It was shown in Ref. [50] that for a polynomial function $p(x)$, the spectral dimension of the operator $p(\partial^2)$ in four-dimensional spacetime is equal to $\frac{4}{N_\mathrm{max}}$, where N_max is the order of the polynomial. From this result the spectral dimension of the cut-off spectral action can be determined as follows. Suppose $h_{\mu\nu}, A_\mu$ and H are small fluctuations of metric, gauge fields and the Higgs field respectively. The heat kernel expansion can be rewritten in the form of the power series of derivatives of these fluctuations i.e.

$$S_b[h_{\mu\nu}, A_\mu, H] \sim \int_M d^4x \sum_{n=0}^{\infty} \left(Q_{n-1} h^{\mu\nu} \left(-\frac{\partial^2}{\Lambda^2} \right)^n h_{\mu\nu} \right.$$
$$\left. + Q'_{n+1} A^\mu \left(-\frac{\partial^2}{\Lambda^2} \right)^{n+1} A_\mu + Q''_n H \left(-\frac{\partial^2}{\Lambda^2} \right)^n H \right) , \tag{3.4.3}$$

where Q_n, Q'_n and Q''_n are proportional to f_n defined in Eq. (3.1.2). Hence,

$$D_s = \lim_{N\to\infty} \frac{4}{N} = 0 \ . \tag{3.4.4}$$

Unlike the cutoff formalism, we will show in this section that the spectral dimension of the zeta spectral action is nontrivial. Since the actions for the Higgs scalar and the gauge fields have the same behavior in the ultraviolet, like in the infrared, their corresponding spectral dimensions coincide with the topological dimension of the manifold and are equal to four. The gravitational spectral dimension can be also defined in a viable way, however such a definition requires some analytical continuation, therefore we elaborate carefully on this point.

The gravitational part of our theory consists of the Weyl square contribution coming from Eq. (3.3.9) and the Ricci scalar R appearing after the renormalisation discussed in the previous section:

$$S_{\rm gr} = \int d^4x \left(\frac{M_{\rm Pl}^2}{16\pi} R - \frac{N_F}{16\pi^2} C_{\mu\nu\eta\xi} C^{\mu\nu\eta\xi} \right) , \tag{3.4.5}$$

where $N_{\rm F}$ is the dimension of the finite Hilbert space.

To compute the spectral dimension one has to extract the quadratic part of $S_{\rm gr}$ for transverse and traceless fluctuations $h_{\mu\nu}$ of the metric tensor $g_{\mu\nu}$, leading to

$$S_{\rm gr} = \frac{M_{\rm Pl}^4}{64\pi} \int d^4x \, h_{\mu\nu} \left[(-\partial^2) - a \left(-\partial^2 \right)^2 \right] h^{\mu\nu} + \mathcal{O}(h^4) \ , \tag{3.4.6}$$

where

$$a := \frac{2N_F}{\pi M_{\rm Pl}^2} \ . \tag{3.4.7}$$

To define the spectral dimension one needs the heat kernel $P(T, x, x')$ corresponding to Eq. (3.4.6), given by

$$K(x, x', T) = \int \frac{d^4p}{(2\pi^4)} e^{ip(x-x')} e^{-\left(p^2 - ap^4\right)T} \ . \tag{3.4.8}$$

Note that, because a is positive the spectrum of $(-\partial^2 - a(-\partial^2)^2)$ is bounded from above, but not bounded from below. Hence, setting $x = x'$, the integral $K(x, x, T)$ diverges, whilst is well-defined for negative a. In spherical coordinates the relevant integral is proportional to

$$\int_0^\infty dp \, p^3 e^{-\left(p^2 - ap^4\right)T} \quad = \quad \frac{1}{8} \, \frac{\left(2\sqrt{-a\,T} \exp\left(\frac{T}{4a}\right) - \sqrt{\pi} \, \mathrm{erf}\left(\frac{1}{2} \frac{\sqrt{-a\,T}}{a} \right) T - \sqrt{\pi} T \right) \mathrm{e}^{-\frac{T}{4a}}}{(-a\,T)^{\frac{3}{2}}} \ , \tag{3.4.9}$$

where the rhs is an analytic function on the complex plane without a ray, that we can choose as a lower half of imaginary axis $[0, -i\infty)$. It means that there exists an *analytic continuation* in a region of positive a; we *define* our integral for positive a as the analytic continuation.

Note that D_s only give meaningful information when it is real, and in our case we will show that the spectral dimension is indeed real. We write

$$\widetilde{D_{\rm s}}(T) = \frac{2\sqrt{-a\,T}(4a+T)\exp\left(\frac{T}{4a}\right) - T(2a+T)\sqrt{\pi}\left[\operatorname{erf}\left(\frac{1}{2}\frac{\sqrt{-a\,T}}{a}\right)+1\right]}{2a\left(2\sqrt{-a\,T}\exp\left(\frac{T}{4a}\right) - \sqrt{\pi}\operatorname{erf}\left(\frac{1}{2}\frac{\sqrt{-a\,T}}{a}\right)T - \sqrt{\pi}\,T\right)}\ , \tag{3.4.10}$$

and plot $\widetilde{D_{\rm s}}(T)$ in Fig. 1. Although we are interested in the limit $T \to 0$, it is worth to note that in the limit $T \to +\infty$ the "running" spectral dimension is also real

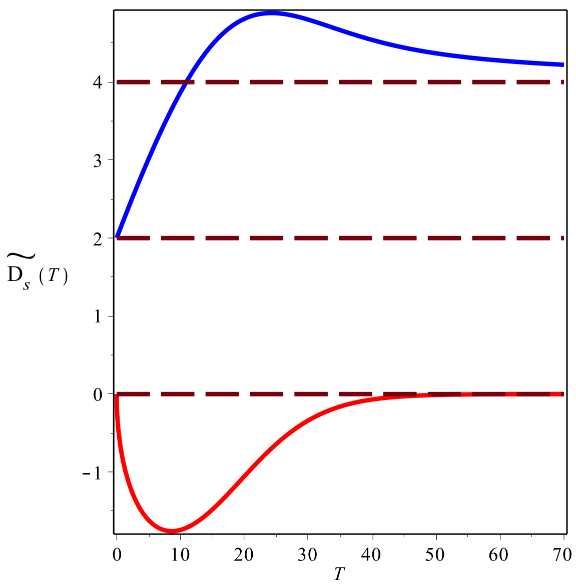

Figure 3.2: *Running spectral dimension* $\widetilde{D_{\rm s}}(T)$ *for* $a = 1$. *The blue line and the red line represent the real and the imaginary part of* $\widetilde{D_{\rm s}}(T)$ *respectively.*

Returning to the *conventional* spectral dimension, we see, that for all *nonzero* real a we get

$$D_{\rm s} \equiv \lim_{T\to 0}\widetilde{D_{\rm s}}(T) = 2\ . \tag{3.4.11}$$

Finally, although in the intermediate range of the parameter T the spectral "running" dimension is imaginary, there exists a sensible "low energy " limit of D_S, valid again for all real a, with

$$D_{\mathrm{s}}^{\mathrm{low}} := \lim_{T\to\infty} \widetilde{D_{\mathrm{s}}}(T) = 4 \ . \tag{3.4.12}$$

Our result Eq. (3.4.11) is in agreement with the fact that the gravitational propagators in our theory decrease faster at infinity, due to the presence of the fourth derivative, thereby improving the ultraviolet convergence of the Feynman loop diagrams. From another point of view our "low energy" result is in agreement with the fact that at very low energies the dynamics does not feel the Weyl square terms.

Remark. *In principle relaxing the normalisation condition discussed in the previous section, one can also renormalise the coefficient in front of the Weyl square action to a positive constant, that would corresponds to negative a in Eq.* (3.4.6). *In this situation, the running spectral dimension $\widetilde{D_{\mathrm{s}}}(T)$ is real for all T, not just at $T = 0$ and $T = \infty$, and the corresponding plot is presented in Fig 3.3.*

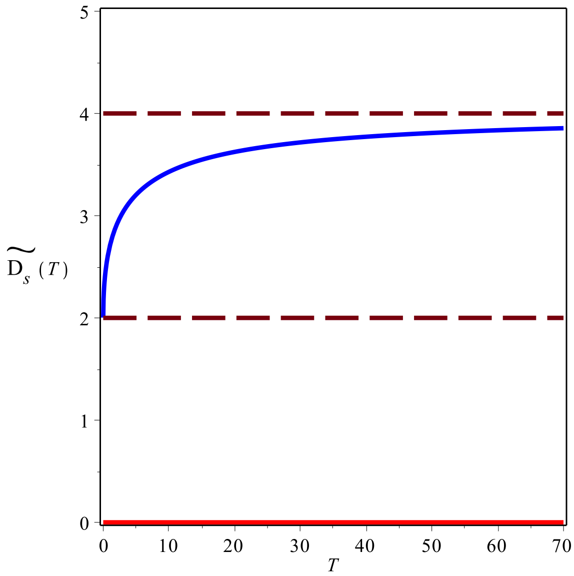

Figure 3.3: *Running spectral dimension* $\widetilde{D_{\rm s}}(T)$ *for* $a = -1$. *The running spectral dimension remains real for all* $T > 0$.

Although the spectral dimension shows improvement in ultraviolet convergence, the presence of fourth derivatives in the theory is well-known for causing linear instability. In the next Chapter we suggest a method that allows one to evade such a problem.

Chapter 4

Linear stability of spectral action

It is clear that the gravitational action (3.4.5) can be derived from both the cut-off spectral action (3.1.3) and the zeta spectral action (3.3.8). One notices that in addition to the Einstein-Hilbert action, the action (3.4.5) carries fourth-order derivatives of the metric tensor. It is well-known result in quantum theory on curved spacetime that the fourth-order derivative terms arise naturally as corrections to the Einstein-Hilbert action [51] at a high energy scale where quantum nature of matter fields manifest. The presence of the fourth-order derivative terms in the spectral actions (3.1.3) and (3.3.8) agrees with fact that the actions are naturally defined at the high energy scale (more precisely, at the approximate unification scale $(10^{14} - 10^{17})$ GeV).

An interesting feature of higher derivative theories are their renormalisability as it was shown in Ref. [52] that the linearised action of (3.4.5) is power-counting renormalisable. Moreover, in the section 3.4 we have seen that the theory has the spectral dimension 2, which is a feature of UV finite or renormalisable quantum gravity models [53]. However, a higher derivative theory can be linearly unstable, which would lead to a nonunitary quantum gravity theory. For the spectral action (3.1.3), the instability is not a serious problem if it is treated as an effective theory i.e. one can introduce some constraints that eliminate unwanted degree of freedom [54, 55] providing that the constraints do not affect the physics below the cut-off scale. In contrast, there is no such an energy scale for zeta spectral action, therefore, imposing a constraint will eliminate a relevant physical information. In this Chapter we show the linear stability of the spectral action on a four-dimensional manifold with torsion and in the absence of any matter fields, adapting the approach proposed in Ref. [5]. We subsequently extend this approach in the nonvacuum case.

The outline of this Chapter is the following. We give a brief background on the Ostrogradski instability, and the Dirac method of solving Hamiltonian constraints in the first two sections. In section 4.3 we add a particular type of torsion into Riemannian geometry and show how to avoid the linear instability in the spectral action. Then in the last section, we use the Dirac method to show that the Hamiltonian of the spectral action on an almost commutative manifold is bounded from below, therefore, the theory does not suffer from linear instability.

4.1 Linear instability

The regularised zeta spectral action, and the asymptotic expansion of the cut-off action (3.1.3) yields the gravitational action of the form

$$S_{\mathrm{gr}}[g_{\mu\nu}] = \int \sqrt{|g|}\left(\bar{\Lambda} + \frac{1}{\kappa^2}R - \alpha_0 ||C||^2\right) d^4x \ , \tag{4.1.1}$$

where $\bar{\Lambda}$ denotes the cosmological constant, $\kappa^2 := 16\pi/M_{\text{Pl}}^2$, α_0 is a positive constant and $||C||^2 := C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$ is the Weyl invariant. Such an action could have an unbounded from below Hamiltonian (and above) which leads to an unnatural behaviour. Suppose we have a system with the unbounded from below Hamiltonian, then by the second law of thermodynamics a state in this system will naturally decay into a lower energy state. However, due to the absence of a ground state, this process could continue endlessly [56]. Moreover, this process will give infinite amount of energy to its surrounding. The following example shows that the 4th-order derivative nondegenerate Lagrangian can give rise to an unbounded Hamiltonian [57].

Example 4.1.1. *Consider a Lagrangian*

$$L(q, \dot{q}, \ddot{q}) = (\ddot{q})^2 F_0(q, \dot{q}) + \ddot{q}F_1(q, \dot{q}) + F_2(q, \dot{q}), \tag{4.1.2}$$

where $F_i(q, \dot{q})$ are smooth function. We consider the case that $L(q, \dot{q}, \ddot{q})$ is nondegenerate, meaning that $F_0(q, \dot{q}) \neq 0$. We define canonical coordinates by

$$\begin{aligned} P_1 &= \frac{\partial L}{\partial \dot{q}} \ , \qquad P_2 = \frac{\partial L}{\partial \ddot{q}} = 2\ddot{q}F_0 + F_1 \ , \\ Q_1 &= q \ , \qquad Q_2 = \dot{q} \ . \end{aligned}$$

Notice that $Q_2 = \dot{Q}_1$ and $\dot{Q}_2 = (P_2 - F_1)/2F_0$, where F_0, F_1 and F_2 are now the *function of Q_1, Q_2. Hence, the Hamiltonian reads*

$$
\begin{aligned}
H &= \sum_{i=1}^{2} P_i \dot{Q}_i - L \\
&= P_1 \dot{Q}_1 + P_2 \dot{Q}_2 - \frac{(P_2 - F_1)^2}{4F_0} - (P_2 - F_1)\frac{F_1}{2F_0} - F_2 \\
&= P_1 Q_2 + P_2 \frac{(P_2 - F_1)}{2F_0} - \frac{(P_2 - F_1)^2}{4F_0} - (P_2 - F_1)\frac{F_1}{2F_0} - F_2 \\
&= P_1 Q_2 + \frac{(P_2 - F_1)^2}{4F_0} - F_2 \ .
\end{aligned} \tag{4.1.3}
$$

By fixing the coordinate Q_1, Q_2 and let P_1 varies, one can see that the Hamiltonian is unbounded from below and above.

One realises that the Lagrangian of the action (4.1.1) is of the form (4.1.2). The equation of motion derived from the linearised action of (4.1.1) has negative energy solutions, therefore, leads to instability [52]. In the next section we will describe the method of solving constrained system, which shall be important for section 4.5.

4.2 Constrained Hamiltonian system

Let M be a four-dimensional globally hyperbolic manifold i.e. $M \cong \mathbb{R} \times \Sigma$, where Σ is a Cauchy surface. Any curve parametrised by $t \in \mathbb{R}$ interesects Σ only once [58]. Consequently, if one picks the time direction along a normal vector on a Cauchy surface, there is no closed timelike curve in the manifold. More importantly, the existence of a Cauchy surface at any instant of time allows us to define the Poisson bracket, which is important operation in the Hamiltonian formalism. Given a dynamical system define by the action

$$S[q] = \int \mathrm{dV} \mathscr{L}(q_i(x), \dot{q}_i(x)), \tag{4.2.4}$$

for q_i is a smooth function (or tensor field), $i \in \{1, ..., N\}$. Passing from Lagrangian to Hamiltonian system, one needs to map the configuration space into the phase space $(q, \dot{q}) \mapsto (q, \pi)$, where $\pi^i := \delta\mathscr{L}/\delta\dot{q}_i$ is the momentum density. Then the Hamiltonian density of the system is defined by the Legendre transformation

$$\mathscr{H}(q, \pi) = \pi^i \dot{q}_i - \mathscr{L}(q, \dot{q}). \tag{4.2.5}$$

However, such a transformation is not always possible i.e. from the definition of momentum density we have

$$\begin{pmatrix} d\pi^i \\ dq_i \end{pmatrix} = \begin{pmatrix} \frac{\delta^2\mathscr{L}}{\delta\dot{q}_j\delta\dot{q}_i} & \frac{\delta^2\mathscr{L}}{\delta q_j\delta\dot{q}_i} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d\dot{q}^i \\ dq_i \end{pmatrix} , \tag{4.2.6}$$

therefore, one cannot replace all $\dot{q}$'s variables with momentum when

$$\det\left(\frac{\delta^2\mathscr{L}}{\delta\dot{q}_j\delta\dot{q}_i}\right) = 0 \ . \tag{4.2.7}$$

Clearly the rank of the matrix is less than N, therefore, the phase space variables are not all independent i.e. there are relations $\phi_k(q, \pi) = 0,\ k < N$ among π's and q's. The relations $\phi_k(q, \pi) = 0$ are called **primary constraints**. The existence of the constraints means that the Hamiltonian of the system cannot be defined on the whole phase space but on a certain subspace defined by the constraints. We define the extended Hamiltonian $\mathscr{H}_{\rm ex}$, which is valid on the whole phase space as

$$\mathscr{H}_{\rm ex} = \mathscr{H} + c^k(q,\pi)\phi_k \ , \tag{4.2.8}$$

where c^k is a Lagrange multiplier. It is clear that $\mathscr{H}_{\rm ex}$ reduces to original Hamiltonian of the system on the surface defined by the primary constraints. Since a constraint is not a new dynamical variable, one requires that any constraint does not evolve in time i.e.

$$\dot{\phi}_k = 0 \ . \tag{4.2.9}$$

The time evolution in the phase space is defined by the extended Hamiltonian, so

$$0 = \dot{\phi}_k(y) = \int d\Sigma_t \{\phi_k(y), \mathscr{H}(x)\} + \int d\Sigma_t c^i \{\phi_k(y), \phi_i(x)\} \ , \tag{4.2.10}$$

where $\{\ ,\ \}$ stands for Poisson bracket, and $d\Sigma_t$ denotes an equal time hypersurface. Eq. (4.2.10) is known as the consistency condition, which allows us to solve the Lagrange multipliers. However, in the case that the lhs of Eq. (4.2.10) does not vanish on the constraint surface, one needs to introduce the **secondary constraint** into the system so that the consistency condition is satisfied.

Any constraint (primary and secondary constraints) extracted from the system can be put into two different types. A constraint is called **first class constraint** if it commutes with all other constraints, otherwise it is called **second class constraint**. The difference between first class constraints and second class constraints can be seen from the solution of Eq. (4.2.10). By substituting a second class constraint into Eq. (4.2.10), one obtains a nonhomogeneous equation, hence some Lagrange multipliers can be determined, while a first class constraints do not give any information about the Lagrange multipliers. The existence of the first class constraint leads to the gauge symmetry of the theory. Fixing the Lagrange multipliers associated with the first class constraints corresponds to choosing a gauge fixing condition; the different choices of Lagrange multipliers yield the same equation of motion. For more examples and the step by step method of solving a constrained Hamiltonian system we refer the reader to Ref. [59].

4.3 Higher derivative gravity as a gauge theory and the spectral action

It has been shown in Ref. [60] that instability could be removed from a higher derivative theory if the theory is treated as a gauge theory: The higher-order equations can be reduced, in the absence of torsion, to the vacuum second-order Einstein's equations. The solutions are conformal equivalence metrics of Ricci-flat spacetimes. Following this approach for the generalised spectral action, we will show that in the absence of torsion the equations of motion combined with the Bianchi identity lead to an integrability condition that implies the reduction to the second-order Einstein's equations.

To generalise the action in Eq. (4.1.1) into a gauge theory with a Poincaré symmetry, one needs to equip a manifold with a tetrad e^a_μ,

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \ , \tag{4.3.11}$$

and a spin connection $\omega_\mu^{ab} \in \mathfrak{so}(1,3)$, satisfying

$$D_\mu e_\nu^a := \partial_\mu e_\nu^a - \Gamma^\alpha_{\mu\nu} e^a_\alpha + \omega_\mu{}^a{}_c e^c_\nu = 0 \ , \tag{4.3.12}$$

where latin characters denote flat spacetime indices, D_μ is the covariant derivative and $\Gamma^\alpha_{\mu\nu}$ is an affine connection. Note that, in this Chapter we choose the signature of the metric to be $(+,-,-,-)$. The curvature two-form of the spin connection, defined by

$$R_{\mu\nu}{}^{ab} := \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega^a_{\mu c}\omega^{cb}_\nu - \omega^a_{\nu c}\omega^{cb}_\mu \ , \tag{4.3.13}$$

is independent of the tetrad basis. In general, the spin connection is not necessarily torsion free. In fact, the curvature two-form (4.3.13) contains a torsion and its derivative. This can be shown by contracting Eq. (4.3.12) with $e^{\nu,b}$:

$$\begin{aligned}
\omega_\mu^{ab} &= e^a_\nu e^{\sigma,b}\Gamma^\nu_{\mu\sigma} + e^a_\nu \partial_\mu e^{\nu,b} \\
&= e^a_\nu e^{\sigma,b}\left(\Gamma^\nu_{(\mu\sigma)} + \Gamma^\nu_{[\mu\sigma]}\right) + e^a_\nu \partial_\mu e^{\nu,b} \\
&= \left(e^a_\nu e^{\sigma,b}\Gamma^\nu_{(\mu\sigma)} + e^a_\nu \partial_\mu e^{\nu,b}\right) + e^a_\nu e^{\sigma,b}\Gamma^\nu_{[\mu\sigma]} \\
&= \left(e^a_\nu e^{\sigma,b}\Gamma^\nu_{(\mu\sigma)} + e^a_\nu \partial_\mu e^{\nu,b}\right) + T^{ab}_\mu \ ,
\end{aligned} \tag{4.3.14}$$

where $T^{ab}_\mu := e^a_\nu e^{\sigma,b} T_\mu{}^\nu{}_\sigma$, for $T_\mu{}^\nu{}_\sigma := \Gamma^\nu_{[\mu\sigma]}$ is the torsion tensor. The subscript notation "( )" denotes symmetrisation $\Gamma^\nu_{(\mu\sigma)} := \frac{1}{2}(\Gamma^\nu_{\mu\sigma} + \Gamma^\nu_{\sigma\mu})$ and "[ ]" denotes antisymmetrisation $\Gamma^\nu_{[\mu\sigma]} := \frac{1}{2}(\Gamma^\nu_{\mu\sigma} - \Gamma^\nu_{\sigma\mu})$.

Defining

$$\omega'^{\ ab}_\mu := e^a_\nu e^{\sigma,b}\Gamma^\nu_{(\mu\sigma)} + e^a_\nu \partial_\mu e^{\nu,b} \ , \tag{4.3.15}$$

we note that ω' is torsion free and the curvature (4.3.13) can be rewritten as

$$R_{\mu\nu}{}^{ab} = R'_{\mu\nu}{}^{ab} + \nabla_\mu T_\nu{}^{ab} - \nabla_\nu T_\mu{}^{ab} + T_\mu{}^a{}_c T_\nu{}^{cb} - T_\nu{}^a{}_c T_\mu{}^{cb} \ , \tag{4.3.16}$$

where ∇ is a covariant derivative acting on a tensor $v_\nu{}^a$ as

$$\nabla_\mu v_\nu{}^a := \partial_\mu v_\nu^a - \Gamma^\alpha_{(\mu\nu)} v_\alpha^a + \omega'^a_\mu{}_c v_\nu^c \ , \tag{4.3.17}$$

and $R'_{\mu\nu}{}^{ab}$ is the curvature two-form of the torsion-free spin connection ω'^{ab}_μ, defined by

$$R'_{\mu\nu}{}^{ab} := \partial_\mu \omega'^{ab}_\nu - \partial_\nu \omega'^{ab}_\mu + \omega'^a_{\mu c}\omega'^{cb}_\nu - \omega'^a_{\nu c}\omega'^{cb}_\mu \ . \tag{4.3.18}$$

Denoting by $\mathcal{T}$ the set of all torsion fields, we consider a particular subset $\mathcal{T}_R \subset \mathcal{T}$, so that the torsion fields $T_\mu{}^{ab} \in \mathcal{T}_R$ satisfy the following properties:

- $T_\mu{}^{ab}$ is antisymmetric in the a, b indices, and hence Eq. (4.3.14) implies that $\omega'_\mu{}^{ab}$ is also antisymmetric in a, b, leading to metric compatibility, and $\omega'_\mu{}^{ab}$ is just the Levi-Civita connection. The reason for choosing totally antisymmetric torsion fields is the following: The general connection on the tangent bundle of a manifold is compatible with the Riemannian metric and has the same geodesics as the Levi-Cevita connection if and only if the connection is the sum of the Levi-Cevita connection and a totally antisymmetric tensor field [61], thus the torsion field is totally antisymmetric.
- $T_\mu{}^{ab}$ yields the curvature tensor with the same symmetric properties as the

Riemmanian curvature tensor, i.e.

$$R_{\mu\nu\sigma\rho} = -R_{\nu\mu\sigma\rho} = R_{\nu\mu\rho\sigma} \ , \tag{4.3.19}$$

$$R_{\mu\nu\sigma\rho} = R_{\sigma\rho\mu\nu} \ , \tag{4.3.20}$$

where $R_{\mu\nu\sigma\rho} = R_{\mu\nu}{}^{ab}\, e_{\sigma,a}\, e_{\rho,b}$. Note that (4.3.19) holds for all torsion fields $T_\mu{}^{ab} \in \mathcal{T}$, while (4.3.20) is only valid for $T_\mu{}^{ab} \in \mathcal{T}_R$. With the above properties of the torsion fields, the Gauss-Bonnet action takes the form we are familiar with in Riemannian geometry, namely

$$\chi_E = \frac{1}{8\pi^2} \int \sqrt{|g|} (R_{\mu\nu}{}^{ab} R^{\mu\nu}{}_{ab} - 4R_\mu{}^a R^\mu{}_a + R^2) d^4x \ . \tag{4.3.21}$$

We note that the above action (4.3.21) is not valid for the more general class of torsions studied in Ref. [62].

Let us also define a traceless tensor $C_{\mu\nu}{}^{ab}$, as

$$C_{\mu\nu}{}^{ab} := \ R_{\mu\nu}{}^{ab} - (e_\mu^{[a} R_\nu{}^{b]} - e_\nu^{[a} R_\mu{}^{b]}) + \frac{1}{3} R e_\mu^{[a} e_\nu^{b]} \ , \tag{4.3.22}$$

where $R_\mu{}^a := \ R_{\mu\nu}{}^{ab} e_b^\nu$ and $R := R_\mu{}^a e_a^\mu$. We can thus generalise the spectral action Eq. (4.1.1) as follows:

$$S_{\text{gr}}[e_\mu^a, \omega_\nu^{ab}] = \int e \left(\bar{\Lambda} + \frac{1}{\kappa^2} R_{\mu\nu}{}^{ab} e_a^\mu e_b^\nu - \alpha_0 C_{\mu\nu}{}^{ab} C^{\mu\nu}{}_{ab} \right) d^4x \ , \tag{4.3.23}$$

where e is defined as $e := \sqrt{|\det(e_\mu^a e_{a,\nu})|} = \sqrt{|g|}$. The action (4.3.23) is not derived from the heat kernel trace, therefore, it is not the spectral action. However, we have shown in the Appendix A that as far as the linearised theory is concerned the action (4.3.23) coincides with the spectral action.

4.4 Stability of the vacuum theory with the cosmological constant

Let us now derive the equations of motion obtained from the generalised action (4.3.23). The variation of the spin connection and the tetrad give respectively,

$$D_\mu C^{\mu\nu}{}_{ab} - \frac{1}{2} T_\mu{}^\nu{}_\alpha C^{\mu\alpha}{}_{ab} = -\frac{1}{4\alpha_0\kappa^2} T_\mu{}^\nu{}_\alpha e^\mu_a e^\alpha_b \ , \tag{4.4.24}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2\alpha_0\kappa^2 \Theta_{\mu\nu} + \frac{\kappa^2}{2}\bar{\Lambda} g_{\mu\nu} \ , \tag{4.4.25}$$

where $\Theta_{\mu\nu} := C_{\mu\alpha}{}^{ab} C_\nu{}^\alpha{}_{ab} - \frac{1}{4} g_{\mu\nu} C_{\rho\sigma}{}^{ab} C^{\rho\sigma}{}_{ab}$. To recover Einstein's equations from Eq. (4.4.25) we need first to set the torsion equal to zero, so that the connection becomes the Levi-Cività one. Thus,

$$\nabla_\mu C'^{\mu\nu}{}_{ab} = 0 \ , \tag{4.4.26}$$

$$R'_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R' = 2\alpha_0\kappa^2 \Theta'_{\mu\nu} + \frac{\kappa^2}{2}\bar{\Lambda} g_{\mu\nu} \ , \tag{4.4.27}$$

where $\Theta'_{\mu\nu} := \Theta_{\mu\nu}|_{T=0}$. Since $\Theta'_{\mu\nu}$ becomes the energy-momentum tensor of the Weyl curvature, and therefore vanishes identically in four dimensions [63], we recover Einstein's equations with a cosmological constant.

The vanishing divergence of the Weyl curvature, Eq. (4.4.26), leads to the integrability condition once combined with the trace of the Bianchi identity

$$\nabla_\mu C'^\mu{}_{\nu\rho\sigma} + (\nabla_\sigma S'_{\nu\rho} - \nabla_\rho S'_{\nu\sigma}) = 0 \ , \tag{4.4.28}$$

where $S'_{\mu\nu} := \frac{1}{2}\left(R'_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R'\right)$ denotes the Schouten tensor. In the basis e^a_μ, we get

$$\nabla_\sigma S'_{\nu\rho} - \nabla_\rho S'_{\nu\sigma} = 0 \ , \tag{4.4.29}$$

which, however, is not the well-known integrability condition. To get the familiar expression [64] we introduce a new basis $e^a_\mu \mapsto \tilde{e}^a_\mu := e^\xi e^a_\mu$, where $\xi(x)$ is a real-value function. Note that the Bianchi identity holds in this new basis, but the covariant derivative of the Weyl tensor transforms as

$$\tilde{\nabla}_\mu \tilde{C'}^\mu{}_{\nu\rho\sigma} = e^{-2\xi}\left(\nabla_\mu C'^\mu{}_{\nu\rho\sigma} + \partial_\mu \xi C'^\mu{}_{\nu\rho\sigma}\right) \ . \tag{4.4.30}$$

To get the integrability condition, we consider Eq. (4.4.28) in the basis $\tilde{e}^a_\mu$ and use Eq. (4.4.30) and the field equation (4.4.26), to obtain

$$\begin{aligned}
0 =& \tilde{\nabla}_\mu \tilde{C'}^\mu{}_{\nu\rho\sigma} + (\tilde{\nabla}_\sigma \tilde{S}'_{\nu\rho} - \tilde{\nabla}_\rho \tilde{S}'_{\nu\sigma}) \\
=& e^{-2\xi}\left(\nabla_\mu C'^\mu{}_{\nu\rho\sigma} + \partial_\mu \xi C'^\mu{}_{\nu\rho\sigma}\right) + (\tilde{\nabla}_\sigma \tilde{S}'_{\nu\rho} - \tilde{\nabla}_\rho \tilde{S}'_{\nu\sigma}) \\
=& (\partial_\mu \xi) e^{-2\xi} C'^\mu{}_{\nu\rho\sigma} + (\tilde{\nabla}_\sigma \tilde{S}'_{\nu\rho} - \tilde{\nabla}_\rho \tilde{S}'_{\nu\sigma}) \\
=& (\partial_\mu \xi) \tilde{C'}^\mu{}_{\nu\rho\sigma} + \tilde{\nabla}_\sigma \tilde{S}'{}_{\nu\rho} - \tilde{\nabla}_\rho \tilde{S}'{}_{\nu\sigma} \ ,
\end{aligned} \tag{4.4.31}$$

where we have used that $e^{-2\xi}C'^\mu{}_{\nu\rho\sigma} = \tilde{C}'^\mu{}_{\nu\rho\sigma}$. Hence, the original manifold is conformally equivalent to a Ricci flat manifold. In other words, there exists a basis $\hat{e}^a_\mu := e^\zeta \tilde{e}^a_\mu$, equal to $\hat{e}^a_\mu = e^{\xi+\zeta} e^a_\mu$ such that

$$\hat{S}'_{\mu\nu} = 0 \ , \tag{4.4.32}$$

leading to a vanishing Ricci tensor, $\hat{R}'_{\mu\nu} = 0$. Therefore, the equation of motion

(4.4.26) is conformally equivalent to the vacuum Einstein equations and the theory is not plagued by a linear instability. Defining $\bar{\chi} := \xi + \zeta$, the Schouten tensor reads

$$\hat{S}'_{\mu\nu} = S'_{\mu\nu} - \nabla_\mu \partial_\nu \bar{\chi} + \partial_\mu \bar{\chi} \partial_\mu \bar{\chi} - \frac{1}{2} g_{\mu\nu} \partial^\alpha \bar{\chi} \partial_\alpha \bar{\chi} \ , \tag{4.4.33}$$

and Eq. (4.4.32) is compatible with Eq. (4.4.27), provided that the scalar field $\bar{\chi}$ satisfies

$$\nabla_\mu \partial_\nu \bar{\chi} - \partial_\mu \bar{\chi} \partial_\nu \bar{\chi} - g_{\mu\nu} \left(\nabla_\alpha \partial^\alpha \bar{\chi} + \frac{1}{2} \partial^\alpha \bar{\chi} \partial_\alpha \bar{\chi} \right) = \frac{1}{4} \kappa^2 \bar{\Lambda} g_{\mu\nu} \ . \tag{4.4.34}$$

In conclusion, considering the variation of the full connection, the higher-order differential equations reduce to Einstein's equations obtained from either Eq. (4.4.26) or from Eq. (4.4.27).

4.5 Hamiltonian of almost commutative spectral action with torsion

Let us now assume that the gravitational action is defined in a four-dimensional globally hyperbolic manifold. Global hyperbolicity also allows us to choose a coordinate system $\{t, x^i\}$ such that the spatial coordinates are orthogonal to the time coordinate, i.e. $g_{ti} = 0$. Let us choose a flat spacetime basis $\{\mathbf{e}^0, \mathbf{e}^I\}$ with $I \in \{1, 2, 3\}$, such that the time direction is preserved:

$$\mathbf{e}^0 = e^0_t dt \quad \text{and} \quad \mathbf{e}^I = e^I_i dx^i \ . \tag{4.5.35}$$

The volume element on an equal time hypersurface is defined by $d\Sigma_t := e^0_t(\det e^I_i)d^3x$. In the previous section we have avoided the linear instability by conformally reducing the equations of motion (4.4.26) to the vacuum Einstein equations. The same method can be extended to the nonvacuum case as long as $\mathcal{L}_{\rm matter}$ is not a function of the spin connection, as for instance for the Lagrangian of a gauge field. Note, however, that there are matter fields whose Lagrangian depends on the spin connection, as for example

$$\mathcal{L}_H = \frac{1}{2}|\nabla'_\mu H|^2 - \frac{1}{12}RH^2 - \mu^2H^2 + \lambda H^4 \ , \tag{4.5.36}$$

$$\mathcal{L}_\psi = i\bar{\psi}(e^\mu_a\gamma^a D_\mu - m)\psi \ , \tag{4.5.37}$$

where $\nabla'_\mu H = \partial_\mu H + [B_\mu, H]$ and $D_\mu\psi := (\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\Sigma_{ab})\psi$, for $\Sigma_{ab} := \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)$. Such Lagrangians lead to the equations of motion

$$\nabla_\mu {C'^{\mu\nu}}_{ab} = \left.\frac{\delta\mathcal{L}_{\rm matter}}{\delta\omega^{ab}_\nu}\right|_{T=0} \neq 0 \ . \tag{4.5.38}$$

In such a case one cannot get the integrability condition using the same trick as previously, and hence one cannot argue the cure of the linear instability following the approach of section 4.4. To show that there is no instability we will check directly that the Hamiltonian is bounded from below.

Without loss of generality, let us turn off the gauge fields and the cosmological constant since they do not depend on the spin connection. By adding the Higgs field and a massive fermionic field into the action (4.3.23), we get

$$S_{\rm gr}[e^a_\mu, \omega^{ab}_\nu] + S_{H,\psi} = \int d^4xe\left(\frac{1}{\kappa^2}{R_{\mu\nu}}^{ab}e^\mu_a e^\nu_b - \alpha_0 {C_{\mu\nu}}^{ab}{C^{\mu\nu}}_{ab} + \mathcal{L}_{H,\psi}\right) \ . \tag{4.5.39}$$

The canonical momenta are

$$\pi^{\beta}_{cd} = -4\alpha_0 C^{t\beta}{}_{cd} + \left(\frac{2}{\kappa^2} - \frac{H^2}{6}\right) e^t_{[c} e^{\beta}_{d]} \ , \tag{4.5.40}$$

$$p^t_0 = 0 \ , \ p^i_I = 0 \ , \tag{4.5.41}$$

where π^{β}_{cd}, p^t_0 and p^i_I stand for the canonical momenta of ω^{cd}_{β}, e^0_t and e^I_i, respectively. Notice that the map $\pi^{ab}_{\beta} \mapsto \partial_t \omega^{ab}_{\beta}$ is not invertible for an arbitrary choice of the spin connection; therefore the Hamiltonian is not well defined. To construct a well-defined Hamiltonian, let us consider a subset of spin connections such that each element can be decomposed into $\omega^{ab}_{\mu} = \Omega^{ab}_{\mu} + \tilde{\omega}^{ab}_{\mu}$ and the following two conditions are satisfied:

$$(i) \ \ C_{\mu\nu}{}^{ab} = \partial_\mu \Omega^{ab}_{\nu} - \partial_\nu \Omega^{ab}_{\mu} + \Omega^{\ a}_{\mu\ c} \Omega^{cb}_{\nu} - \Omega^{\ a}_{\nu\ c} \Omega^{cb}_{\mu} \ . \tag{4.5.42}$$

$$(ii) \ \ (\Omega^{ac}_{[\mu} \tilde{\omega}_{\nu]c}^{\ \ b} + \tilde{\omega}_{[\mu}{}^{ac} \Omega_{\nu]c}^{\ \ b}) e^{\mu}_b e^{\nu}_a = 0 \ . \tag{4.5.43}$$

We will call (i) and (ii) the "splitting conditions," since they make the scalar curvature independent of $\Omega_{\mu}{}^{ab}$. To see this we rewrite the curvature $R = R_{\mu\nu}{}^{ab} e^{\mu}_a e^{\nu}_b$ in terms of Ω and $\tilde{\omega}$. Thus,

$$\begin{aligned} R_{\mu\nu}{}^{ab} = {} & \partial_\mu \Omega^{ab}_{\nu} - \partial_\nu \Omega^{ab}_{\mu} + \Omega^{\ a}_{\mu\ c} \Omega^{cb}_{\nu} - \Omega^{\ a}_{\nu\ c} \Omega^{cb}_{\mu} \\ & + \partial_\mu \tilde{\omega}^{ab}_{\nu} - \partial_\nu \tilde{\omega}^{ab}_{\mu} + \tilde{\omega}^{\ a}_{\mu\ c} \tilde{\omega}^{cb}_{\nu} - \tilde{\omega}^{\ a}_{\nu\ c} \tilde{\omega}^{cb}_{\mu} \\ & - 2(\Omega^{ac}_{[\mu} \tilde{\omega}_{\nu]c}^{\ \ b} + \tilde{\omega}_{[\mu}{}^{ac} \Omega_{\nu]c}^{\ \ b}) \ . \end{aligned} \tag{4.5.44}$$

Assuming the validity of the conditions $i)$ and $ii)$ above, the scalar curvature reads

$$\begin{aligned} R &= R_{\mu\nu}{}^{ab} e^\mu_a e^\nu_b \\ &= C_{\mu\nu}{}^{ab} e^\mu_a e^\nu_b + (\partial_\mu \tilde{\omega}_\nu^{ab} - \partial_\nu \tilde{\omega}_\mu^{ab} + \tilde{\omega}_\mu{}^a{}_c \tilde{\omega}_\nu^{cb} - \tilde{\omega}_\nu{}^a{}_c \tilde{\omega}_\mu{}^{cb}) e^\mu_a e^\nu_b \\ &\quad - 2(\Omega^{ac}_{[\mu} \tilde{\omega}_{\nu]c}^{\ b} + \tilde{\omega}_{[\mu}{}^{ac} \Omega_{\nu]c}^{\ b}) e^\mu_a e^\nu_b \\ &= (\partial_\mu \tilde{\omega}_\nu^{ab} - \partial_\nu \tilde{\omega}_\mu^{ab} + \tilde{\omega}_\mu{}^a{}_c \tilde{\omega}_\nu^{cb} - \tilde{\omega}_\nu{}^a{}_c \tilde{\omega}_\mu{}^{cb}) e^\mu_a e^\nu_b \ . \end{aligned} \tag{4.5.45}$$

Note that the considered subset of spin connections is not empty, since it contains connections of all conformal Ricci flat geometry. Moreover, the splitting conditions hold automatically in the linearised theory.

Proposition 4.5.1. *The splitting conditions hold in any linearised metric gravitational theory*

Proof. To prove this statement, let $h_{\mu\nu}$ denote the metric perturbation. The condition (ii) is clearly satisfied since $\Omega_\mu{}^a{}_c \tilde{\omega}_\nu{}^{cb} e^\mu_a e^\nu_b$ is of order higher than $O(h^2)$. For condition (i) one chooses the transverse traceless metric perturbation $\bar{h}_{\mu\nu}$ which satisfies the Laplace equation

$$\Box \bar{h}_{\mu\nu} = 0 \ , \tag{4.5.46}$$

where $\Box$ denotes the flat-space d'Alembertian. The Weyl tensor is

$$\begin{aligned} C_{\mu\nu\sigma\rho} =& \frac{1}{2}(\partial_\sigma \partial_\nu \bar{h}_{\mu\rho} + \partial_\rho \partial_\mu \bar{h}_{\nu\sigma} - \partial_\rho \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \partial_\mu \bar{h}_{\nu\rho}) \\ =& \eta_{\mu\lambda} \partial_\sigma \bar{\Gamma}^\lambda_{\nu\rho} - \eta_{\mu\lambda} \partial_\rho \bar{\Gamma}^\lambda_{\nu\sigma} \ , \end{aligned} \tag{4.5.47}$$

where $\bar{\Gamma}^\lambda_{\nu\rho} := \frac{1}{2}\eta^{\lambda\mu}(\partial_\nu \bar{h}_{\rho\mu} + \partial_\rho \bar{h}_{\nu\mu} - \partial_\mu \bar{h}_{\nu\rho})$. Then using the definition of the spin connection, one can rewrite the Weyl tensor in terms of the derivative of $\Omega_\mu{}^{ab}$, and therefore the condition (i) is satisfied. $\square$

Defining

$$\Pi^{\beta}_{cd} := \frac{\partial \mathcal{L}}{\partial(\partial_t \Omega^{cd}_{\beta})} \quad \text{and} \quad \tilde{\pi}^{\beta}_{cd} := \frac{\partial \mathcal{L}}{\partial(\partial_t \tilde{\omega}^{cd}_{\beta})} \ , \tag{4.5.48}$$

where $\mathcal{L}$ is the Lagrangian density of the action (4.5.39), and assuming the splitting conditions, one can then show that

$$\Pi^{\beta}_{cd} = -4\alpha_0 C^{t\beta}{}_{cd} \ , \tag{4.5.49}$$

$$\tilde{\pi}^{\beta}_{cd} = 2(\frac{1}{\kappa^2} - \frac{H^2}{12}) e^t_{[c} e^{\beta}_{d]} \ . \tag{4.5.50}$$

From the definition of the canonical momentum, we get the constraints $\Pi^t_{cd} = 0$ and $\tilde{\pi}^t_{cd} = 0$, which are primary first-class constraints and can be solved using the gauge-fixing conditions $\Omega_t{}^{ab} = 0,\ g^{ij} D_i \Omega_j{}^{ab} = 0$ and $\tilde{\omega}_t{}^{ab} = 0,\ g^{ij} D_i \tilde{\omega}_j{}^{ab} = 0$. The remaining constraints

$$\phi^t_0 := p^t_0 = 0 \ , \tag{4.5.51}$$

$$\phi^i_I := p^i_I = 0 \ , \tag{4.5.52}$$

$$\phi_c := \Pi^i_{cd} e^d_i = 0 \ , \tag{4.5.53}$$

$$\varphi^j_c := \Pi^j_{dc} e^d_t - 4\alpha_0 C^{ji}{}_{cd} e^d_i = 0 \ , \tag{4.5.54}$$

$$\phi^j_{cd} := \tilde{\pi}^j_{cd} - 2(\frac{1}{\kappa^2} - \frac{H^2}{12}) e^t_{[c} e^j_{d]} = 0 \ . \tag{4.5.55}$$

are primary second-class constraints, and are also obtained from the definition of the canonical momentum.

In what follows, let P, Q stand for the canonical variables and the symbol "$\approx$" denote the equality holding on the surface spanned by all constraints, called the "constraint surface" for short. Imposing all constraints, the Hamiltonian reads

$$
\begin{aligned}
\mathscr{H} &= P_I \partial_t Q^I - \mathcal{L} \\
&= \Pi^i_{cd} \partial_t \Omega^{cd}_i + \tilde{\pi}^i_{cd} \partial_t \tilde{\omega}^{cd}_i + p^\beta_c \partial_t e^c_\beta + p_H \dot{H} + p_\psi \dot{\psi} - \mathcal{L} \\
&\approx -\frac{1}{8\alpha_0} \Pi^i_{cd} \Pi^{cd}_i + \alpha_0 C^{ij}{}_{cd} C_{ij}{}^{cd} - \left(\frac{1}{\kappa^2} - \frac{H^2}{12} \right) R_{ij}{}^{cd} e^i_c e^j_d + \mathscr{H}_{H,\psi} \\
&\approx -\frac{1}{4\alpha_0} \Pi^i{}_{0I} \Pi_i{}^{0I} + \alpha_0 C^{ij}{}_{IJ} C_{ij}{}^{IJ} - \left[\frac{1}{8\alpha_0} \Pi^i_{IJ} \Pi^{IJ}_i - 2\alpha_0 C^{ij}{}_{0I} C_{ij}{}^{0I} \right] \\
&\quad - \left(\frac{1}{\kappa^2} - \frac{H^2}{12} \right) R_{ij}{}^{cd} e^i_c e^j_d + \mathscr{H}_{H,\psi} \\
&\approx -\frac{1}{4\alpha_0} \Pi^i{}_{0I} \Pi_i{}^{0I} + \alpha_0 C^{ij}{}_{IJ} C_{ij}{}^{IJ} - \left(\frac{1}{\kappa^2} - \frac{H^2}{12} \right) R_{ij}{}^{cd} e^i_c e^j_d + \mathscr{H}_{H,\psi} \ , \qquad (4.5.56)
\end{aligned}
$$

where p_H and p_ψ are the canonical momenta of the scalar field and the fermion field, respectively. Note that the term $\left[\frac{1}{8\alpha_0} \Pi^i_{IJ} \Pi^{IJ}_i - 2\alpha_0 C^{ij}{}_{0I} C_{ij}{}^{0I} \right]$ vanishes due to the symmetry (4.3.20) of the curvature tensor.

Denote the set of primary second-class constraints by $\Phi^A := \{\phi^t_0, \phi^i_I, \phi_c, \varphi^j_c, \phi^j_{cd}\}$ and define a new Hamiltonian density as

$$
\mathscr{H}_{\text{ex}} := \mathscr{H} + u_A \Phi^A, \qquad (4.5.57)
$$

where u_A are Lagrange multipliers. By imposing the consistency condition on the constraints $\phi_c, \varphi^j_c, \phi^t_0$ and ϕ^i_I one obtains the secondary constraint (the full details can be found in Appendix C)

$$
\begin{aligned}
\chi &:= \frac{1}{2\alpha_0} \Pi^k_{0I} \Pi^{0I}_k + 2\alpha_0 C_{ij}{}^{lk} C^{ij}{}_{lk} + i\bar{\psi} \left(\gamma^I e^i_I D_i \psi - 2m\psi \right) - 2\mu^2 H^2 + 2\lambda H^4 \\
&= 0 \ . \qquad (4.5.58)
\end{aligned}
$$

Using the constraint (4.5.58) the Hamiltonian reads

$$
\begin{aligned}
\mathscr{H} &\approx 2\alpha_0 C^{ijlk}C_{ijlk} - \frac{1}{2}i\bar{\psi}\gamma^I e^i_I D_i\psi - (\frac{1}{\kappa^2} - \frac{H^2}{12})R_{ij}{}^{IJ}e^i_I e^j_J + \frac{1}{2}g_{tt}p_H^2 - \frac{1}{2}g^{ij}\partial_i H^\dagger \partial_j H \\
&\approx \mathscr{H}_{C^2} + \mathscr{H}_{\mathrm{GR}} \ ,
\end{aligned} \tag{4.5.59}
$$

where $\mathscr{H}_{C^2}$ and $\mathscr{H}_{\mathrm{GR}}$ are defined, respectively, as

$$
\mathscr{H}_{C^2} := 2\alpha_0 C^{ijlk}C_{ijlk} - \frac{1}{2}i\bar{\psi}\gamma^I e^i_I D_i\psi \ , \tag{4.5.60}
$$

$$
\mathscr{H}_{\mathrm{GR}} := -(\frac{1}{\kappa^2} - \frac{H^2}{12})R_{ij}{}^{IJ}e^i_I e^j_J + \frac{1}{2}g_{tt}p_H^2 - \frac{1}{2}g^{ij}\partial_i H^\dagger \partial_j H \ . \tag{4.5.61}
$$

Theorem 4.5.2. *The Hamiltonian density $\mathscr{H}_{C^2} + \mathscr{H}_{\mathrm{GR}}$ is bounded from below iff the torsion vanishes*

Proof. The Hamiltonian density $\mathscr{H}_{C^2}$ is bounded from below, since the first term is positive definite and the second one is proportional to the Hamiltonian of a massless fermion. To show that $\mathscr{H}_{\mathrm{GR}}$ is also bounded from below, let us recall the gauge-fixing condition $\omega_t{}^{ab} = 0$, which implies $T_t{}^{ab} = 0$, since torsion is independent of the Levi-Cività spin connection. Using Eq. (4.3.16) we deduce that

$$
\begin{aligned}
R_{ti}{}^{0I} &= R'_{ti}{}^{0I} + \nabla_t T_i{}^{0I} - \nabla_i T_t{}^{0I} + T_t{}^0{}_J T_i{}^{JI} - T_i{}^0{}_J T_t{}^{JI} \\
&= R'_{ti}{}^{0I} \ ,
\end{aligned} \tag{4.5.62}
$$

while for $T_\mu{}^{ab} \in \mathcal{T}_R$, the scalar curvature obtained by contracting Eq. (4.3.16) reads

$$
R = R' - ||T||^2 \ . \tag{4.5.63}
$$

Hence,

$$
\begin{aligned}
R_{ij}{}^{IJ} e^i_I e^j_J &= R - 2R_{ti}{}^{0I} e^t_0 e^i_I \\
&= R' - ||T||^2 - 2R'_{ti}{}^{0I} e^t_0 e^i_I \\
&= (R' - 2R'_{ti}{}^{0I} e^t_0 e^i_I) - (3T_t{}^{IJ} T^t{}_{IJ} + T_i{}^I{}_K T_j{}^{KJ} e^i_I e^j_J) \\
&= R'_{ij}{}^{IJ} e^i_I e^j_J - T_i{}^I{}_K T_j{}^{KJ} e^i_I e^j_J \ ,
\end{aligned} \tag{4.5.64}
$$

which implies that $\mathscr{H}_{\rm GR}$ can be rewritten as

$$
\mathscr{H}_{\rm GR} \approx -(\frac{1}{\kappa^2} - \frac{H^2}{12}) R'^{IJ}_{ij} e^i_I e^j_J + \frac{1}{2} g_{tt} p_H^2 - \frac{1}{2} g^{ij} \partial_i H^\dagger \partial_j H + (\frac{1}{\kappa^2} - \frac{H^2}{12}) T_i{}^I{}_K T_j^{KJ} e^i_I e^j_J \ , \tag{4.5.65}
$$

where a prime $'$ referring to torsion-free quantities. Assuming the Higgs field does not exceed the Planck mass, i.e. $H^2 < 12/\kappa^2$, and noting that for the metric of signature $(+,-,-,-)$,

$$
\begin{aligned}
T_{ijk} T^{ijk} =& g^{il} g^{jm} g^{kn} T_{ijk} T_{lmn} \\
=& \sum_{i,j,k=1}^{3} g^{ii} g^{jj} g^{kk} T_{ijk} T_{ijk} \leq 0 \ ,
\end{aligned} \tag{4.5.66}
$$

one concludes that the last term on the rhs of Eq. (4.5.65), which can be written as $(\frac{1}{\kappa^2} - \frac{H^2}{12}) T_{ijk} T^{ljk}$, is negative definite, and therefore, unbounded from below. In contrast, the first two terms on the rhs of Eq. (4.5.65) are just the canonical Hamiltonian of the Palatini action in the presence of a scalar field interaction term [65], leading to the classical dynamics of the Einstein-Hilbert action in the presence of the scalar field. We hence conclude that $\mathscr{H}_{\rm GR}$ is bounded from below if and only if torsion vanishes. $\square$

Finally, let us check whether the result agrees with section 4.3. In the vacuum case the constraint (4.5.58) becomes

$$\chi := \frac{1}{2\alpha_0}\Pi^k_{0I}\Pi^{0I}_k + 2\alpha_0 C_{ij}{}^{lk}C^{ij}{}_{lk} = 0 \ . \tag{4.5.67}$$

Since $\Pi^k_{0I}\Pi^{0I}_k$ and $C_{ij}{}^{lk}C^{ij}{}_{lk}$ are positive definite, the constraint (4.5.67) implies that both terms have to vanish, and hence the Hamiltonian reads

$$\mathscr{H} \approx \mathscr{H}_{\mathrm{GR}} \approx -\frac{1}{\kappa^2}R_{ij}{}^{cd}e^i_c e^j_d \ . \tag{4.5.68}$$

Hence, the Hamiltonian does not depend on the Weyl tensor, in agreement with the fact that the vacuum case reduces to Einstein gravity. Clearly, then, this Hamiltonian will give the same dynamics as Einstein's equations in vacuum.

The above analysis can be easily applied in the spectral action. In the simple vacuum case and considering a torsion field $T_\mu{}^{ab} \in \mathcal{T}_R$, the third-order differential equations can be reduced to the second-order Einstein equations. Therefore, in this case the theory does not suffer from a linear instability. In the case of an almost commutative torsion geometry and considering only matter fields whose Lagrangians do not depend on the spin connection, one can still guarantee the stability of the theory employing the method discussed in the section 4.4. Moreover, if fermions and conformal invariant scalar fields are present, the linear stability will still hold, provided that the splitting conditions (4.5.42) and (4.5.43) are satisfied.

Chapter 5

Dirac operator and dispersion relation

The definition of spectral triples is based on the structure of Riemannian manifolds, while the relativistic physics is defined on Lorentzian ones. However, the translation between Riemannian geometry to Lorentzian geometry is not clear in the framework of spectral geometry, due to two reasons. The first reason is that there is no complete notion of Lorentzian spectral triple. Although there are a few practical notions of Lorentzian spectral triple [66, 67, 68, 69], a proof of the reconstruction theorem, analogous to that given in Ref. [1] is still missing. The second reason stems from the nonellipticity of the Dirac operators on pseudo-Riemannian manifolds. The Laplacian derived from the Dirac operator on pseudo-Riemannian manifold, has an unbounded spectrum (below and above), therefore, the heat kernel trace is ill-defined, leaving a difficulty in defining a bosonic spectral action.

Despite the incomplete knowledge of the Lorentzian spectral triple, we argue that there is a notion of Lorentzian spectral triple that allows one to derive the energy-momentum dispersion of a fermion on Minkowski spacetime in a purely geometric manner. The aim of this Chapter is twofold. First we investigate whether a particular definition of Lorentzian spectral triple gives suitable description of causal structure on Minkowski spacetime. Second, we establish the link between the energy-momentum dispersion relation and the geometry of an almost commutative spacetime.

This Chapter will be organised as follows. We introduce the definition of Lorentzian spectral triple in section 5.1, which leads to the causal structure given in section 5.2. The causal structure allow us to define a function that served as a squared of Lorentzian distance in almost commutative geometry. This function allows us to identify the causal structure of the almost commutative geometry with a Minkowski spacetime. Then, in section 5.3, we classify spinors traveling along geodesics as causal, harmonic and noncausal spinors, and show that only harmonic spinors are allowed to travel along null geodesics. Furthermore, we show that the eigenvalues of harmonic spinor satisfies the energy-momentum dispersion relation. In the last section, we consider a toy model with inner fluctuated Dirac operator and show that the result obtained from section 5.3, is still valid.

5.1 Lorentzian spectral triple

Although noncommutative geometry has been applied to a relativistic theory like the Standard Model, the definition of a Lorentzian spectral triple remains an open question, the reason mainly being the lack of manifold reconstruction theorem analogous to Connes' reconstruction theorem for a commutative spectral triple [1]. Nevertheless, there are a few definitions of Lorentzian spectral triples in the literature [66, 67, 68, 69]. In this Chapter we adopt the definition proposed by [66], which will allow us to define a causal structure. Moreover, for a commutative case that is constructed from a globally hyperbolic manifold, one can define a distance formula (which will be defined in the next section) similar to the spectral distance formula. The Lorentzian version of spectral distance formula was propose in [70], it was proved that the formula leads to the geodesic distance in Minkowski space.

Definition 5.1.1. *Lorentzian spectral triple* [66]
A Lorentzian spectral triple is given by $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{H}, \mathcal{D}, \mathcal{J})$, *where*

- $\mathcal{A}$ *is a nonunital dense* $*$*-subalgebra of a* C^**-algebra, and* $\tilde{\mathcal{A}}$ *its unitisation*
- $\mathcal{H}$ *is a Krein space with an indefinite product* $(\cdot, \cdot)$
- $\mathcal{J}$ *is a bounded self-adjoint symmetry operator,* $\mathcal{J} = \mathcal{J}^*$, $\mathcal{J}^2 = 1$, *commuting with* $\mathcal{A}$. *The role of* $\mathcal{J}$ *– dubbed as* fundamental symmetry *or* signature operator *– is to turn the Krein space* $\mathcal{H}$ *into a Hilbert space. Note that,* $\mathcal{H}_{\mathcal{J}}$ *is the same space as* $\mathcal{H}$ *with inner product* $\langle \cdot, \cdot \rangle := (\cdot, \mathcal{J} \cdot)$, *hence a Hilbert space.*
- $\mathcal{D}$ *is a densely defined operator on* $\mathcal{H}_{\mathcal{J}}$ *such that*
  - $\mathcal{D} = -\mathcal{J}\mathcal{D}^*\mathcal{J} =: -\mathcal{D}^+$ *i.e. it is Krein anti-self-adjoint on* $\mathcal{H}$ *(Note that* $\mathcal{D}^*$ *denotes the adjoint with respect to* $\mathcal{H}_{\mathcal{J}}$*)*
  - $\forall a \in \tilde{A}$, $[\mathcal{D}, a]$ *extends to a bounded operator on* $\mathcal{H}_{\mathcal{J}}$
  - $\forall a \in \mathcal{A}$, $a(1 + \langle \mathcal{D} \rangle)^{-1/2}$ *is compact on* $\mathcal{H}_{\mathcal{J}}$, *where* $\langle \mathcal{D} \rangle^2 := \frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D})$.

- *there exists a densely defined self-adjoint operator* $\mathcal{T}$ *with* $\mathrm{Dom}D \cap \mathrm{Dom}\mathcal{T}$ *dense in* $\mathcal{H}_{\mathcal{J}}$ *such that*
  - $(1+\mathcal{T}^2)^{-1/2} \in \tilde{\mathcal{A}}$
  - $\mathcal{J} = -N[\mathcal{D}, \mathcal{T}]$ *for some positive element* $N \in \tilde{\mathcal{A}}$.

The example of such a Lorentzian spectral triple is given by [71]

$$(C_0^\infty(M), C_b^\infty(M), L^2(M,S), -i\nabla\!\!\!\!/\,)\ , \tag{5.1.1}$$

where M is a four-dimensional globally hyperbolic Lorentzian spin manifold with signature $(-,+,+,+)$, $C_0^\infty(M)$ is the algebra of smooth functions vanishing at infinity, and $C_b^\infty(M)$ is for the space of smooth bounded functions on the manifold. The Krein $L^2(M,S)$ is the space of square integrable smooth sections of the spinor bundle. The Dirac operator is defined by $-i\nabla\!\!\!\!/ := -i\gamma^\mu\nabla_\mu$, where ∇_μ is the spin connection on M. Note that, using the sign convention $(-,+,+,+)$, we choose the representation of gamma matrices such that $(\gamma^0)^* = -\gamma^0$, $(\gamma^i)^* = \gamma^i$, with the anti-commutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}_4\ , \tag{5.1.2}$$

where μ, ν are spacetime indices.

The fundamental symmetry $\mathcal{J}$ can be derived from the lapse function N and the global time function $\mathcal{T}$, as follows: For a globally hyperbolic Lorentzian manifold M, there exists a global smooth time function $\mathcal{T}$ on M such that the line element of the manifold M splits as

$$ds^2 = -Nd\mathcal{T}^2 + ds^2_{\mathcal{T}}\ , \tag{5.1.3}$$

where $ds^2_{\mathcal{T}}$ is the line element on the Cauchy hypersurface $\Sigma_{\mathcal{T}}$ at constant time $\mathcal{T}$ and N is the lapse function. The fundamental symmetry in terms of N and $\mathcal{T}$ is $\mathcal{J} = -N[\mathcal{D}, \mathcal{T}]$; a condition that guarantees the Lorentzian signature.

To include a causal structure into the algebra, one defines a set of real-valued functions which are nondecreasing along a future-directed causal curve:

$$\mathcal{C} = \{f \in C_b^\infty(M) : \; f(x) \leq f(y) \text{ iff } x \preceq y, \; \forall x, y \in M\} \; . \tag{5.1.4}$$

The set $\mathcal{C}$ is called the **causal cone** and its elements are **causal functions**. In a globally hyperbolic spacetime (M, g), the geodesic distance coincides with the Lorentzian distance function [72]

$$d(x,y) = \inf \left\{ f(y) - f(x) \middle| \; f \in \mathcal{C} \; , \; \text{ess sup } g(\nabla f, \nabla f) \leq -1 \right\}, \;\; \forall x, y \in M \text{ with } x \preceq y \; . \tag{5.1.5}$$

In the following, we highlight the definition of the causal cone expressed in terms of the spectral triple [70, 71].

Proposition 5.1.2. *Let $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{H}, \mathcal{D}, \mathcal{J})$ be a commutative Lorentzian spectral triple constructed from a globally hyperbolic manifold. Then $f \in \tilde{\mathcal{A}}$ is a causal function iff*

$$(\psi, [\mathcal{D}, f]\psi) \leq 0 \; , \; \forall \psi \in \mathcal{H} \; .$$

Note that, Proposition 5.1.2 will allow us to generalise the definition of causal cone to noncommutative cases. Furthermore, in Minkowski spaces, one can rewrite Eq. (5.1.5) using purely spectral data.

Proposition 5.1.3. *Let $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{H}, \mathcal{D}, \mathcal{J})$ be an even commutative Lorentzian spec-*

tral triple with the grading operator γ *constructed from an even dimensional Minkowski space. For every* $x, y \in \mathcal{M}$*, the distance function*

$$\tilde{d}(p,q) := \inf_f \{\max\{0, f(p) - f(q)\} \mid \forall \psi \in \mathcal{H}, (\psi, \mathcal{J}[\mathcal{D}, f]\psi) \leq -(\psi, i\gamma\psi)\}$$

agrees with the usual Lorentzian distance.

For simplicity, let us consider a Minkowski spacetime, denoted by $\mathcal{M}$, as the globally hyperbolic spacetime. In a four-dimensional Minkowski spacetime, any two points $x, y \in \mathcal{M}$ can be connected by a spacelike curve, i.e. a curve $\gamma : [0,1] \to \mathcal{M}$ such that $g(\dot{\gamma}, \dot{\gamma}) > 0$ along the curve. However, some of these points can also be connected by a causal curve, i.e. $g(\dot{\gamma}, \dot{\gamma}) \leq 0$ everywhere along the curve; these points are called causally related and are denoted by $x \preceq$ y.

Consider two points x, y in the Minkowski four-dimensional spacetime $\mathcal{M}$ connecting through a curve γ. We define the **extremal length square** as

$$L^2(x,y) := \begin{cases} -\sup\{\ l(\gamma)^2 := \left(\int_\gamma \sqrt{-g(\dot{\gamma},\dot{\gamma})}d\tau\right)^2 \mid g(\dot{\gamma},\dot{\gamma}) \leq 0\} & ,\ x \preceq y \\ \\ \sup\{\ l(\gamma)^2 := \left(\int_\gamma \sqrt{g(\dot{\gamma},\dot{\gamma})}d\tau\right)^2 \mid g(\dot{\gamma},\dot{\gamma}) > 0\} & ,\ x \npreceq y\ . \end{cases} \tag{5.1.6}$$

Since Minkowski spacetime is flat, $L^2(x,y) = -(x_0 - y_0)^2 + \|\mathbf{x} - \mathbf{y}\|^2$, which is zero or negative for two causally related points and strictly positive otherwise. Notice that the distance defined by

$$d(x,y) = \begin{cases} \sqrt{-L^2(x,y)} & , \ x \preceq y \\ \\ 0 & , \ x \npreceq y \end{cases} \tag{5.1.7}$$

vanishes for both space-like and light-like separation. Hence, by using the definition (5.1.6) above, we can differentiate between points which are connected by a null geodesic and those which are not causally related.

5.2 Causal structure and distance

In the previous section we have seen that the commutative Lorentzian spectral triple $(C_0^\infty(\mathcal{M}), C_b^\infty(\mathcal{M}), L^2(S,\mathcal{M}), -i\partial\!\!\!/)$, yields a spectral distance equivalent to the geodesic distance for Minkowski spacetime. Next, we shall define a distance function for an almost commutative geometry, namely the product of this Lorentzian spectral triple with a finite spectral triple, and then examine the implications of the proposed distance function

Consider a two sheet space, defined by the tensor product of a commutative Lorentzian spectral triple and a discrete spectral triple $(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F)$, as in Eq. (2.2.35). Following Ref. [66], one can then define a causal structure on the space of states $\mathcal{S}(\tilde{\mathcal{A}})$ of the two sheet space, using only the spectral data of the almost commutative manifold; we highlight the procedure below.

Definition 5.2.1. *Let* $\mathcal{C} = \{a \in \tilde{\mathcal{A}} \mid a = a^*, \ (\psi, [\mathcal{D}, a]\psi) \leq 0, \forall \psi \in \mathcal{H}\}$. *Two* *states $\omega, \omega' \in \mathcal{S}(\tilde{\mathcal{A}})$ are causally related i.e. $\omega \preceq \omega'$ iff for any $a \in \mathcal{C}$, one has*

$$\omega(a) \leq \omega'(a). \tag{5.2.8}$$

In this case $\mathcal{P}(\tilde{\mathcal{A}})$ is the union of $\mathcal{M}_0 := \mathcal{M} \times \{0\}$ and $\mathcal{M}_1 := \mathcal{M} \times \{1\}$, hence the name of two sheet spacetime. Thus, one may think of having two sheets of four-dimensional Minkowski spacetimes embedded in a five-dimensional one. Since we are interested in the causal relation between points on $\mathcal{M}_0$ and $\mathcal{M}_1$, we consider a particular type of mixed states $\omega_{x,\xi} \in \mathcal{N}(\tilde{\mathcal{A}}) := \mathcal{M} \times [0,1] \subset \mathcal{S}(\tilde{\mathcal{A}})$ defined by

$$\omega_{x,\xi}(a \oplus b) = \xi a(x) + (1-\xi) b(x), \tag{5.2.9}$$

for $a, b \in C_0^\infty(\mathcal{M})$. Such states $\omega_{x,\xi}$ can be considered as covering the area between the two sheets. Note that a pure state can be recovered with $\xi = 0$ or $\xi = 1$.

Theorem 5.2.2. *The two states $\omega_{x,\xi},\ \omega_{y,\eta} \in \mathcal{N}(\mathcal{A})$ are causally related if and only if $x \preceq y$ on $\mathcal{M}$ and*

$$l(\gamma) \geq \frac{|\arcsin\sqrt{\eta} - \arcsin\sqrt{\xi}|}{|m|} , \tag{5.2.10}$$

where $l(\gamma)$ represents the length of a causal curve γ going from x to y on the manifold $\mathcal{M}$.

The above theorem [66] implies that if the discrete Dirac operator is trivial, i.e. $m = 0$, the causal relation holds only when $\xi = \eta$. If $m \neq 0$, any two points $(x, 0) \in \mathcal{M}_0$ and $(y, 1) \in \mathcal{M}_1$ are causally related iff there is a causal curve γ connecting x and y such that

$$l(\gamma) \geq \frac{\pi}{2|m|} , \tag{5.2.11}$$

which implies

$$-\sup_{\gamma} l^2(\gamma) + \frac{\pi^2}{4|m|^2} \leq 0 \ . \tag{5.2.12}$$

Definition 5.2.3. *For any* $(x,i),(y,j) \in \mathcal{M} \times \{0,1\}$ *with* $i,j \in \{0,1\}$ *we define the* **extremal length squared** $L_m^2 : (\mathcal{M} \times \{0,1\}) \times (\mathcal{M} \times \{0,1\}) \to \mathbb{R}$, *as follows*

$$L_m^2[(x,i),(y,j)] = \begin{cases} \frac{4}{\pi^2} L^2(x,y) + \frac{1}{|m|^2} & , \ i \neq j \\ \\ \frac{4}{\pi^2} L^2(x,y) & , \ i = j \end{cases} \tag{5.2.13}$$

From Eq. (5.1.6), we see that the above defined function is negative semi-definite when the points (x,i) and (y,j) are causally related, and positive otherwise.

Combining the definition (5.2.13) above and Theorem 5.2.2, one obtains a criterion for any two points (pure states) to be causally related.

Proposition 5.2.4. *The pure states* $(x,i),(y,j)$*, defined on an almost commutative manifold, are said to be causally related if and only if* $x \preceq y$ *on* $\mathcal{M}$ *and*

$$L_m^2[(x,i),(y,j)] \leq 0 \ . \tag{5.2.14}$$

We will refer to the above condition as the **causal structure**.

In analogy to (5.1.7), the distance on the two-sheet space can be defined as

$$d[(x,i),(y,j)] = \begin{cases} \sqrt{-L_m^2[(x,i),(y,j)]} & , \ (x,i) \preceq (y,j) \\ \\ 0 & , \ (x,i) \npreceq (y,j) \end{cases} \tag{5.2.15}$$

One notices that, both causal and geometrical structure of the two-sheet space is exactly the same as the one of a pair of four-dimensional Minkowski spacetimes embedded in a five-dimensional one ($\mathcal{M}_5 := \mathcal{M} \times [0,1]$) with $1/|m|$ denoting the separation between the two four-dimensional manifolds. The metric of the five-dimensional Minkowski spacetime $\mathcal{M}_5$ reads

$$g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1/|m|^2 \end{pmatrix} , \tag{5.2.16}$$

where μ, ν are the spacetime indices in Minkowski spacetime, which being flat is denoted by $\eta_{\mu\nu}$. The metric (5.2.16) can be seen as a Wick-rotated version of (2.2.34).

Using metric (5.2.16), any two points in the two-sheet spacetime are causally related provided they are causally related in $(\mathcal{M}_5, g)$. The line element in $\mathcal{M}_5$ is

$$\begin{aligned} ds^2 = g_{ab}dx^a dx^b &= \eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{|m|^2}dx_F^2 \\ &= ds_{\mathcal{M}}^2 + ds_F^2 \ , \end{aligned} \tag{5.2.17}$$

where dx_F is the infinitesimal of the interval $[0,1]$.

Making the appropriate choice for the Dirac operator $\mathcal{D}_5$ in $\mathcal{M}_5$, such that

$$\mathcal{D}_5^2 = -\nabla\!\!\!/^2 - |m|^2 \frac{\partial^2}{\partial x_F^2} \ , \tag{5.2.18}$$

the spectral distance expression (5.1.5) for a globally hyperbolic manifold, implies the geodesic expression as the one derived from the metric (5.2.16). To specify our notation, let us remark that $\mathcal{D}_5$ is defined by Eq. (5.2.18), whereas $\mathcal{D}$ will refer to the Dirac operator as defined for an almost commutative manifold.

The Lorentzian version of the spectral distance formula is still applicable on the two sheet space. Note that, to recover the $\mathcal{D}^2$ operator as defined for an almost commutative Lorentzian manifold, one chooses the boundary condition for a spinor in a five-dimensional Minkowski space such that for any $\phi \in L^2(\mathcal{M}_5, S)$

$$(\mathcal{D}_5^2\phi)\Big|_{\mathcal{M}\times\{0,1\}} = \mathcal{D}^2\phi\Big|_{\mathcal{M}\times\{0,1\}} = (-\nabla\!\!\!/^2 + |m|^2)\phi\Big|_{\mathcal{M}\times\{0,1\}}. \tag{5.2.19}$$

Remark. *Unlike the Euclidean case, one should not jump to the conclusion that $\mathcal{D}_5^2$ is the inverse of ds^2. Since $ds^2 = 0$ for a null curve, there is no notion of $(ds^2)^{-1}$ on such curve.*

5.3 Dirac operator and dispersion relation

In this section we will investigate the relation between distance for a two-sheet space and Dirac operator. To proceed, let divide spinors into three different classes

Definition 5.3.1. *Let $x \in \mathcal{M}$ and a spinor field $\psi \in L^2(\mathcal{M}) \otimes \mathbb{C}^2$. For $\psi^\dagger\psi \neq 0$, ψ*

*is **causal** at x if*

$$\frac{\psi^\dagger \mathcal{D}^2 \psi}{\psi^\dagger \psi}\Big|_x \geq 0 \ , \tag{5.3.20}$$

*and is **harmonic** if the equality holds. Otherwise, the spinor is **noncausal**.*

Let us note that in this study we restrict ourselves to the case of harmonic spinors, the reason being that we want to investigate their implications for the dispersion relation. The next proposition will show that harmonic spinors yield the energy-momentum dispersion relation, meaning that they can be interpreted as physical matter fields.

Proposition 5.3.2. *Let X be a compact subset*[1] *of $\mathcal{M}$, and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the product of the Lorentzian spectral triple $(C^\infty(X), L^2(X,S), -i\partial\!\!\!/)$ and the finite spectral triple $(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F)$. The eigenspinors Ψ_n of the Dirac operator, with $\Psi_n^\dagger \Psi_n \neq 0$, are harmonic iff their eigenvalues satisfy the energy-momentum dispersion relation.*

Proof

Let $\Psi_n := \psi_p \otimes e_i \in \mathrm{Dom}\mathcal{D}$ be a normalised eigenspinor of $\mathcal{D}$, where ψ_p and e_i are eigenstates of $-\partial\!\!\!/^2$ and $\mathcal{D}_F^2$, respectively.

$$\begin{aligned}
\Psi_n^\dagger \mathcal{D}^2 \Psi_n =& (-\psi^\dagger \nabla\!\!\!\!/^2 \psi_p) e_i^\dagger e_i + \psi^\dagger \psi (e_i^\dagger \mathcal{D}_F^2 e_i) \\
=& \ (E^2 - \mathbf{p}^2) \psi^\dagger \psi_p e_i^\dagger e_i - m_i^2 \psi^\dagger \psi_p e_i^\dagger e_i \\
=& \ (E^2 - \mathbf{p}^2 - m_i^2) \Psi_n^\dagger \Psi_n \ ,
\end{aligned} \tag{5.3.21}$$

where $-E^2$ denotes the eigenvalue of the $\partial^2/\partial t^2$ operator, and $-p_i^2$ stands for the eigenvalue of $\partial^2/\partial x_i^2$ ($\mathbf{p}$ denotes a three-vector).Correspondingly, the r.h.s. of

[1] We choose the compact set $X \subset \mathcal{M}$ so that $\psi_p = \xi_p e^{i(-Et+\mathbf{p}\cdot\mathbf{x})}$, for ξ_p a constant spinor, is square integrable.

Eq. (5.3.21) is the energy-momentum dispersion relation for a massive fermion iff Ψ_n *is harmonic.*

One may argue that the energy-momentum dispersion relation has its origin in the geometric construction of the almost commutative manifold. Suppose a massive spinor is traveling between $(x, 0)$ and $(y, 1)$, one can show that these two points are causally related. To perform computation we will assume that the spinor travel through $\mathcal{M}_5$ along a curve $(\gamma(t), x_F(t))$, by Proposition 5.3.2 the spinor's velocity at each point on the curve is given by

$$v = \left(\frac{E}{m_i}, \frac{\mathbf{p}}{m_i}, 1\right)$$

If the spinor is causal, then using metric 5.2.16

$$g_{ab}v^a v^b = \eta_{\mu\nu}v^\mu v^\nu + \frac{1}{|m_i|^2} \geq 0 \ ,$$

where $v^\mu := \left(\frac{E}{m_i}, \frac{\mathbf{p}}{m_i}\right)$, then we have

$$l(\gamma) = \int_0^1 \sqrt{-\eta_{\mu\nu}v^\mu v^\nu} dt \geq \int_0^1 \frac{1}{|m_i|} dt = \frac{1}{|m_i|} \ . \tag{5.3.22}$$

Which tell us that $(x, 0)$ and $(y, 1)$ are causally related (the factor $\pi/2$ was omitted from 5.2.16 otherwise it would appear here as well).

Due to the causal relation between the two sheets, one may interpret this statement as the interaction between a fermion on one sheet and an anti-fermion on the other one. Furthermore, in the two-sheet spacetime one can treat massive and massless fermions in an equal footing i.e. if one defines $p_4 := m_i$, then

$$E^2 - \mathbf{p}^2 - p_4^2 = 0 \ . \tag{5.3.23}$$

Since we identify $\mathcal{D}_F$ with the spatial part of the Dirac operator in $\mathcal{M}_5$, an eigenvalue of $\mathcal{D}_F$ is indeed the momentum of a fermion. Hence, one can think of a massive fermion in the two-sheet space as a massless fermion in the five-dimensional spacetime.

To highlight the validity of Proposition 5.3.2 in the case of inner fluctuations of the Dirac operator, we will consider below a simple toy model, namely electroweak theory with massless neutrinos.

5.4 A toy model: Electroweak theory with massless neutrinos

Consider the electroweak theory and assume neutrinos to be massless. To explain this theory in the context of almost commutative spectral geometry, let us take the product of a Lorentzian spectral triple $(C_0^\infty(\mathcal{M}), L^2(\mathcal{M}, S), -i\partial\!\!\!/)$ with a finite spectral triple for the electroweak theory [4]. The spectral triple for the discrete (internal) space F is given by the algebra $\mathcal{A}_F$, the Hilbert space $\mathcal{H}_F$ and the Dirac operator $\mathcal{D}_F$:

$$\begin{aligned}
\mathcal{A}_F &= \mathbb{C} \oplus \mathbb{H} , && (5.4.24) \\
\mathcal{H}_F &= \mathcal{H}_l \oplus \mathcal{H}_{\bar{l}} , && (5.4.25) \\
\mathcal{D}_F &= \begin{pmatrix} 0 & Y^* & 0 & 0 \\ Y & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{Y}^* \\ 0 & 0 & \bar{Y} & 0 \end{pmatrix} , && (5.4.26)
\end{aligned}$$

where Y is a 2×2 mass matrix

$$Y = \begin{pmatrix} 0 & 0 \\ 0 & m_e \end{pmatrix} , \tag{5.4.27}$$

with m_e a complex parameter.

Assuming all inner fluctuations to vanish, apart from those of the scalar field Φ, the fluctuated Dirac operator for the almost commutative manifold is

$$\mathcal{D}_\Phi = -i\partial\!\!\!/ \otimes \mathbb{I}_F + \gamma^5 \otimes \Phi , \tag{5.4.28}$$

with

$$\begin{aligned}
\Phi =& \mathcal{D}_F + a[\mathcal{D}_F, b] + J_F a[\mathcal{D}_F, b] J_F^* \\
=& \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix} , && (5.4.29)
\end{aligned}$$

for $a, b \in C_0^\infty(\mathcal{M}, A_F)$ and

$$\phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{m}_e h_2 & \bar{m}_e(h_1+1) \\ 0 & -m_e \bar{h}_2 & 0 & 0 \\ 0 & m_e(\bar{h}_1+1) & 0 & 0 \end{pmatrix}, \tag{5.4.30}$$

where h_1, h_2 are complex functions. The trace of Φ^2 is given by

$$\mathrm{Tr}\Phi^2 = 2|m_e|^2|\varphi|^2, \tag{5.4.31}$$

where $\varphi := (h_1+1, h_2)$ is a doublet. Assuming φ undergoes symmetry breaking and denoting by v the new VEV, we can choose $\varphi = (v+h, 0)$, where h is a small fluctuation around the vacuum.

To derive the dispersion relation, we will need $\mathcal{D}_\Phi^2$, given by

$$\begin{aligned} \mathcal{D}_\Phi^2 = & -\partial\!\!\!/^2 \otimes \mathbb{I}_F - i\gamma^\mu\gamma^5 \otimes \partial_\mu\Phi - i\gamma^5\gamma^\mu \otimes \partial_\mu\Phi + \mathbb{I}_4 \otimes \Phi^2 \\ = & -\partial\!\!\!/^2 \otimes \mathbb{I}_F + \mathbb{I}_4 \otimes \Phi^2 \ , \end{aligned} \tag{5.4.32}$$

where we have used $\{\gamma^5, \gamma^\mu\} = 0$. We denote the basis of $\mathcal{H}_l$ and $\mathcal{H}_{\bar{l}}$ by $\{\nu_R, e_R, \nu_L, e_L\}$ and $\{\bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L\}$, respectively.

The dispersion relation associated with harmonic eigenspinors $\psi_p \otimes e_L$ and $\psi_p \otimes \nu_L$ (the same result can be obtain for right-handed particles and anti-particles) can be derived as follows:

$$(\psi_p \otimes e_L, \mathcal{D}_\Phi^2 \psi_p \otimes e_L) = 0 \ . \tag{5.4.33}$$

However,

$$
\begin{aligned}
(\psi_p \otimes e_L, \mathcal{D}_\Phi^2 \psi_p \otimes e_L =&(\psi_p, \partial\!\!\!/^2 \psi_p)(e_L, e_L) + (\psi_p, \psi_p)(e_L, \Phi^2 e_L) \\
=&(-E^2 + \mathbf{p}^2)(\psi_p, \psi_p)(e_L, e_L) + \|m_e\|^2(v^2 + 2vh + h^2)(\psi_p, \psi_p)(e_L, e_L) \\
=&-E^2 + p^2 + \|m_e\|^2(v^2 + 2vh + h^2)(\psi_p, \psi_p)(e_L, e_L)\ .
\end{aligned}
\tag{5.4.34}
$$

Hence,

$$
E^2 = p^2 + \|m_e\|^2(v^2 + 2vh + h^2)\ . \tag{5.4.35}
$$

Since the fluctuation is small, we have $E^2 \sim p^2 + \|m_e\|^2 v^2$, which corresponds to proposition 5.3.2. Similarly, the harmonic spinor $\psi_p \otimes \nu_L$ yields

$$
E^2 = p^2\ , \tag{5.4.36}
$$

corresponding to the case a) of the proof in proposition 5.3.2.

Chapter 6

Conclusions

Noncommutative spectral geometry is a theoretical framework that offers a purely geometric explanation for the Standard Model of particle physics. We have addressed several open issues regarding dynamics of bosons and fermions on almost commutative manifolds.

We explored the dynamics of bosons on almost commutative spectral geometry. We introduced a new definition of the bosonic spectral action using the zeta function regularisation; the new definition does not use any external input, such as a cutoff function or a cutoff scale. The corresponding theory is local, unitary and renormalisable. The spectral dimensions for fields of various spin are non trivial and the theory is ultraviolet complete. The zeta spectral action is an interesting alternative to the usual cutoff spectral action. We have argued that it has the same predictive power to the cut-off spectral action. In addition, the way it treats the fundamental scales could also shed some light on the explanation of some fundamental questions.

The gravitational sector of the cut-off spectral action, and the regularised zeta spectral action contains higher-derivative terms, hence, one may wonder whether these gravitational theories may be plagued by linear instabilities, namely the appearance of negative energy modes. We have addressed this question here in two steps. We have first considered the simple vacuum case and shown that by introducing a particular type of torsion, one can apply the method presented in Ref. [60] and reduce the fourth-order differential equations in those of second-order derived from vacuum general relativity, if and only if the torsion field vanishes. We have then considered the spectral action of an almost commutative torsion geometry. For this latter case we have shown that one cannot obtain the integrability condition in the presence of either fermion fields or scalar fields. We have, however, argued that there exists a class of almost commutative torsion geometry that leads to a Hamiltonian which is bounded from below and hence argued that the theory does not suffer from a local instability.

Although there is no bosonic spectral action on a Lorentzian manifold, the fermionic spectral action is well-defined. This allows one to study the dynamics of fermions on the product of Lorentzian spectral triple and a finite spectral triple. We investigate how the causal structure on an almost commutative Lorentzian geometry give rise to the energy-dispersion relation of fermions. The study is inspired by the fact that the spectral distance between a pair of pure states in $M \times F$ was shown to be related to the infinitesimal distance ds^2 between two points in M and the distance between internal states in F, via the Pythagorean theorem [27]. Such a relation was shown [14] also to be valid for $1/ds^2$. For the latter case, one may observe a similarity between the Pythagorean theorem and the energy-momentum dispersion relation, implying a geometric origin of the dispersion relation.

To confirm the above observation, one has to reformulate the inverse distance, given by the inverse of the Dirac operator, in the context of Lorentzian almost commutative spectral geometry. Following Ref. [66], one can write down the spectral triple for a Lorentzian almost commutative manifold, and get the corresponding Dirac operator.

Having the Lorentzian Dirac operator we are able to calculate the distance for a two-sheet manifold and define the notion of a causal structure for such a geometry. We were then able to show that the causal structure on a flat almost commutative space can be identified with the causal structure on the five-dimensional Minkowski space with metric

$$g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1/|m|^2 \end{pmatrix} .$$

We have then suggested that spinors may be classified into causal, harmonic and noncausal ones. The condition satisfied by the harmonic spinors propagating in an almost commutative manifold is equivalent to the causal relation given in Ref. [66]. We have further shown that a spinor is harmonic if and only if it satisfies the energy-momentum dispersion relation. We have hence shown the geometric origin of the dispersion relation in the context of almost commutative spectral geometry.

The predictive power of the regularised zeta spectral action can be improved if the exact values of dimensionful couplings of lower dimensional operators can be determined. We have seen that such operators cannot be generated within classical geometries, since the process requires renormalisation. However, one may consider the zeta function on quantum geometries such as fuzzy geometries, quantum groups etc. Calculating the zeta spectral action on various quantum geometries may help us determine the values of these dimensionful couplings.

We have seen that an appropriate Lorentzian spectral triple can give a good physical description of fermions. This may lead to the definition of bosonic spectral action on the Lorentzian spectral triple. Since the fermionic spectral action is well-defined on a Lerentzian spectral triple, based on the idea presented in Ref. [39, 40, 41], the bosonic spectral action may be generated as a conformal anomaly of the Lorentzian fermionic spectral action. As a future research topic, it is interesting to use the quantisation procedure suggested in Ref. [10].

Appendix A

Equivalence of linearised actions

Let ${T_\mu}^{ab} \in \mathcal{T}_R$, the linearised theory obtained from the action (4.3.23) is equivalent to the linearised spectral action with torsion defined in Ref. [62, 73]. In particular, the action (4.3.23) is linearly stable if and only if the spectral action with torsion is linearly stable.

To show this suppose $T_{\mu\nu\sigma} \in \mathcal{T}_R$, we have by definition that

$$0 = R_{\mu\nu\rho\sigma} - R_{\rho\sigma\mu\nu} = (dT)_{\mu\nu\rho\sigma} - 2\left(\nabla_\rho T_{\sigma\mu\nu} + \nabla_\sigma T_{\rho\mu\nu}\right) \ , \tag{A.0.1}$$

and

$$0 = R_{\mu\rho} - R_{\rho\mu} = g^{\nu\sigma}\left(R_{\mu\nu\rho\sigma} - R_{\rho\sigma\mu\nu}\right) = 2\nabla_\sigma {T^\sigma}_{\rho\mu} \ . \tag{A.0.2}$$

Hence, the spectral action (modulo the Euler characteristic number) is reduced to

$$\begin{aligned} S_{\mathrm{TS}} \sim & f_4\Lambda^4 a_0(\mathcal{D}^2) + f_2\Lambda^2 a_2(\mathcal{D}^2) + f(0)a_4(\mathcal{D}^2) \\ \sim & \int \sqrt{|g|}d^4x\left(\alpha_2\Lambda^4 + \frac{1}{\kappa^2}(R' - ||T||^2) - \alpha_0||C'||^2\right) \ , \end{aligned} \tag{A.0.3}$$

(note that the torsion tensor $T_{\mu\nu\sigma} := 3\tilde{T}_{\mu\nu\sigma}$, where $\tilde{T}_{\mu\nu\sigma}$ denotes the torsion defined in Ref. [62]). To compare $S_{\rm gr}$ with $S_{\rm TS}$, we will write explicitly the torsion terms which are contained in $S_{\rm gr}$. Consider the square of the traceless tensor $C_{\mu\nu}{}^{ab}$ defined in Eq. (4.3.22):

$$
\begin{aligned}
||C||^2 =& ||R_{\mu\nu\rho\sigma}||^2 - 2||R_{\mu\nu}|| + \frac{1}{3}R^2 \\
=& ||R'_{\mu\nu\rho\sigma}||^2 + \frac{1}{4}||dT||^2 - \frac{1}{3}R'||T||^2 + 4B(T) + \frac{1}{3}||T||^4 \\
&- 2\left(||R'_{\mu\nu}|| + \frac{1}{3}||T||^4 - \frac{1}{2}R'||T||^2 + 2B(T)\right) \\
&+ \frac{1}{3}\left(R'^2 - 2R'||T||^2 + ||T||^4\right) \\
=& ||C'||^2 + \frac{1}{4}||dT||^2 \ ,
\end{aligned}
\tag{A.0.4}
$$

where $B(T) := -R'_{\mu\nu}T^{\mu\sigma\rho}T^{\nu}{}_{\sigma\rho} + \frac{1}{4}R'||T||^2$ and the curvature scalar R is $R = R' - ||T||^2$. Substituting $||C||^2$ and R in the action (4.3.23) we get

$$
S_{\rm gr} = \int \sqrt{|g|}\left[\alpha_2\Lambda^4 + \frac{1}{\kappa^2}\left(R' - ||T||^2\right) - \alpha_0\left(||C'||^2 + \frac{1}{4}||dT||^2\right)\right] d^4x \ . \tag{A.0.5}
$$

Using Eq. (A.0.1) we rewrite $||dT||^2$ as

$$
\begin{aligned}
||dT||^2 &= (dT)_{\mu\nu\rho\sigma}(dT)^{\rho\sigma\mu\nu} \\
&= 4(-\nabla_\rho T_{\sigma\mu\nu} + \nabla_\sigma T_{\rho\mu\nu})(-\nabla^\mu T^{\nu\rho\sigma} + \nabla^\nu T^{\mu\rho\sigma}) \\
&= 16\nabla_\rho T_{\sigma\mu\nu}\nabla^\mu T^{\nu\rho\sigma} \\
&= 16\nabla^\mu(T^{\nu\rho\sigma}\nabla_\rho T_{\sigma\mu\nu}) - 16T^{\nu\rho\sigma}\nabla^\mu\nabla_\rho T_{\sigma\mu\nu} \\
&= 16\nabla^\mu(T^{\nu\rho\sigma}\nabla_\rho T_{\sigma\mu\nu}) + 16T^{\nu\rho\sigma}\nabla_\rho\nabla^\mu T_{\sigma\mu\nu} - 16T^{\nu\rho\sigma}[\nabla^\mu, \nabla_\rho]T_{\sigma\mu\nu} \\
&= 16\nabla^\mu(T^{\nu\rho\sigma}\nabla_\rho T_{\sigma\mu\nu}) - 32T^{\nu\rho\sigma}\left(R'^{\mu}_{\ \rho\sigma\alpha} - \frac{1}{2}\delta^\mu_\sigma R'_{\rho\alpha}\right)T^{\alpha}_{\ \mu\nu} \ . \qquad \text{(A.0.6)}
\end{aligned}
$$

Note that to obtain the last line, we have used the fact that the divergence of torsion field vanishes (Eq. (A.0.2)) and the identity

$$
[\nabla_\mu, \nabla_\nu]V_{\rho\sigma\alpha} = R'^{\ \ \ \beta}_{\mu\nu\rho}V_{\beta\sigma\alpha} + R'^{\ \ \ \beta}_{\mu\nu\sigma}V_{\rho\beta\alpha} + R'^{\ \ \ \beta}_{\mu\nu\alpha}V_{\rho\sigma\beta} \ . \qquad \text{(A.0.7)}
$$

Thus, on a manifold without boundary, the action $S_{\rm gr}$ reads

$$
\begin{aligned}
S_{\rm gr} &= \int\sqrt{|g|}\left[\alpha_2\Lambda^4 + \frac{1}{\kappa^2}\left(R' - ||T||^2\right) - \alpha_0||C'||^2\right]d^4x \\
&\quad +8\alpha_0\int\sqrt{|g|}T^{\nu\rho\sigma}\left(R'^{\mu}_{\ \rho\sigma\alpha} - \frac{1}{2}\delta^\mu_\sigma R'_{\rho\alpha}\right)T^{\alpha}_{\ \mu\nu}d^4x \\
&= S_{\rm ST} + 8\alpha_0\int\sqrt{|g|}T^{\nu\rho\sigma}\left(R'^{\mu}_{\ \rho\sigma\alpha} - \frac{1}{2}\delta^\mu_\sigma R'_{\rho\alpha}\right)T^{\alpha}_{\ \mu\nu}d^4x \ . \qquad \text{(A.0.8)}
\end{aligned}
$$

Since the terms in the integrand appearing on the rhs of Eq. (A.0.8) are of order $\mathcal{O}(\omega^3)$, they can be discarded in the linearised theory. Thus, the actions $S_{\rm gr}$ and $S_{\rm TS}$ lead to theories which are equivalent in linear order (similar argument can show that $S_{\rm gr}$ and $S_{\rm ST}$ give the same linearised theory as the spectral action of

Ref. [73]).

Appendix B

Solving Hamiltonian contraints

Let us make a remark that will be useful later. Denoting by Φ^A the set of primary second-class constraints, one has

$$\begin{aligned} 0 \approx \dot{\Phi}^A &= \{\Phi^A, e\mathscr{H}_{\text{ex}}\} = \{\Phi^A, e\mathscr{H} + eu_B\Phi^B\} \\ &= \{\Phi^A, e\mathscr{H}\} + eu_B\{\Phi^A, \Phi^B\} + u_B\{\Phi^A, e\}\Phi^B \\ &\approx e\left(\frac{1}{e}\{\Phi^A, e\mathscr{H}\} + u_B\{\Phi^A, \Phi^B\}\right) \ , \end{aligned} \tag{B.0.1}$$

where u_B stand for Lagrange multipliers. Hence, if the quantity $\left(\frac{1}{e}\{\Phi^A, e\mathscr{H}\} + u_B\{\Phi^A, \Phi^B\}\right)$ is weakly equal to zero, then the consistency condition is satisfied.

In what follows we will derive the constraint (4.5.58). Note that we use the identities

$$\delta e^\mu_a = -\, e^\mu_b e^\nu_a \delta e^b_\nu \ , \tag{B.0.2}$$

$$\delta e = e e^\mu_a \delta e^a_\mu = -e e^a_\mu \delta e^\mu_a \ . \tag{B.0.3}$$

Let us first reduce the number of unknown Lagrange multipliers by imposing the consistency condition on the constraints $\phi_c = 0$ and $\varphi^j_c = 0$.

- $0 \approx \dot{\phi}_c = \{\phi_c, e\mathscr{H}_{\text{ex}}\}$:

Using Eq. (B.0.1) the consistency condition implies

$$0 \approx \{\phi_c, \mathscr{H}\} + u^0_t\{\phi_c, \phi^t_0\} + u^I_i\{\phi_c, \phi^i_I\} + w^a_j\{\phi_c, \varphi^j_a\} \ . \tag{B.0.4}$$

Contraction with $e^c_t = (e^0_t, 0, 0, 0)$ then yields

$$\begin{aligned} 0 &\approx \{e^c_t\phi_c, \mathscr{H}\} + u^0_t e^c_t\{\phi_c, \phi^t_0\} + u^I_i e^c_t\{\phi_c, \phi^i_I\} + w^a_j e^c_t\{\phi_c, \varphi^j_a\} \\ &\approx \{e^0_t\phi_0, \mathscr{H}\} + u^0_t\{e^0_t\phi_0, \phi^t_0\} - u^0_t\phi_0\{e^0_t, \phi^t_0\} + u^I_i\{e^0_t\phi_0, \phi^i_I\} + w^j_a e^0_t\{\phi_0, \varphi^a_j\} \\ &\approx \{e^0_t\phi_0, \mathscr{H}\} + u^0_t\{e^0_t\phi_0, \phi^t_0\} + u^I_i\{e^0_t\phi_0, \phi^i_I\} + w^a_j e^0_t\{\phi_0, \varphi^j_a\} \ . \end{aligned} \tag{B.0.5}$$

- $0 \approx \dot{\varphi}^j_J = \{\varphi^j_J, e\mathscr{H}_{\text{ex}}\}$:

$$0 \approx \{\varphi^j_J, \mathscr{H}\} + u^0_t\{\varphi^j_J, \phi^t_0\} + u^I_i\{\varphi^j_J, \phi^i_I\} - u_c\{\phi_c, \varphi^j_J\} \ . \tag{B.0.6}$$

Contraction with e^J_j then yields

$$0 \approx \{e^J_j\varphi^j_J, \mathscr{H}\} + u^0_t\{e^J_j\varphi^j_J, \phi^t_0\} + u^I_i\{e^J_j\varphi^j_J, \phi^i_I\} - u_c e^J_j\{\phi_c, \varphi^j_J\} \ . \tag{B.0.7}$$

Combining Eqs. (B.0.5),(B.0.7) and using $e^J_j\varphi^j_J = e^0_t\phi_0$, one gets

$$u^c e^J_j = -w^J_j e^c_t \ . \tag{B.0.8}$$

Defining the scalar $C := \frac{1}{3} w^J_j e^j_J$ one then obtains

$$u^a = -C e^a_t \ , \qquad w^J_j = C e^J_j \ . \tag{B.0.9}$$

As a consequence of (B.0.9) the total Hamiltonian is reduced to

$$\begin{aligned}\mathscr{H}_{\text{ex}} &= \mathscr{H} - C e^a_t \phi_a + C e^J_j \varphi^j_J + u^0_t \phi^t_0 + u^I_i \phi^i_I + u^{ab}_j \phi^j_{ab} \\ &= \mathscr{H} + u^0_t \phi^t_0 + u^I_i \phi^i_I + u^{ab}_j \phi^j_{ab}.\end{aligned} \tag{B.0.10}$$

Next, to obtain the constraint Eq. (4.5.58), we analyze the consistency of the constraints $\phi^t_0 = 0$ and $\phi^i_I = 0$.

- $0 \approx \dot{\phi}^t_0 = \{\phi^t_0, e\mathscr{H}_{\text{ex}}\}$:

We have

$$\begin{aligned}0 \approx& \frac{1}{e}\{\phi^t_0, e\mathscr{H}\} + u^{ab}_j \{p^t_0, \phi^j_{ab}\} \\ \approx& \ \{p^t_0, \mathscr{H}\} + \frac{1}{e}\mathscr{H}\{p^t_0, e\} - 2u^{0J}_j \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right)(e^t_0)^2 e^j_J \\ \approx& \ \frac{1}{2\alpha_0} e^t_0 \Pi^i_{0K} \Pi^{0K}_i + \{p^t_0, \mathscr{H}_{H,\psi}\} - e^t_0 \mathscr{H} - 2u^{0J}_j \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right)(e^t_0)^2 e^j_J \\ \approx& \ \frac{3}{4\alpha_0} e^t_0 \Pi^i_{0K} \Pi^{0K}_i - \alpha_0 e^t_0 C_{ij}{}^{IJ} C^{ij}{}_{IJ} + \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right) e^t_0 R^{IJ}_{ij} e^i_I e^j_J \\ &+ \left(\{p^t_0, \mathscr{H}_{H,\psi}\} - e^t_0 \mathscr{H}_{H.\psi}\right) - 2u^{0J}_j \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right)(e^t_0)^2 e^j_J \ .\end{aligned} \tag{B.0.11}$$

Multiplying the above equation, Eq. (B.0.11), with e_t^0 we obtain

$$
\begin{aligned}
0 \approx\; & \frac{3}{4\alpha_0}\Pi^i_{0K}\Pi^{0K}_i - \alpha_0 C_{ij}{}^{IJ}C^{ij}{}_{IJ} + \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right) R^{IJ}_{ij} e^i_I e^j_J \\
& + \left(e^0_t\{p^t_0, \mathscr{H}_{H,\psi}\} - \mathscr{H}_{H.\psi}\right) - 2u^{0J}_j \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right) e^t_0 e^j_J \ . \qquad \text{(B.0.12)}
\end{aligned}
$$

- $0 \approx \dot{\phi}^k_K = \{\phi^k_K, e\mathscr{H}_{\text{ex}}\}$:

We have

$$
\begin{aligned}
0 \approx & \frac{1}{e}\{p^k_K, e\mathscr{H}\} + u^{ab}_j\{p^k_K, \phi^j_{ab}\} \\
\approx\; & \{p^k_K, \mathscr{H}\} + \frac{1}{e}\mathscr{H}\{p^k_K, e\} - 2u^{0J}_j \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right) e^t_0 e^k_J e^j_K \\
\approx\; & \frac{1}{2\alpha_0}\Pi^k_{0I}\Pi^{0I}_j e^j_K + 4\alpha_0 e^m_K C^{kl}{}_{IJ} C_{ml}{}^{IJ} - 2\left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right) R_{ij}{}^{IJ} e^i_I e^k_J e^j_K \\
& + \{p^k_K, \mathscr{H}_{H,\psi}\} - e^k_K \mathscr{H} - 2u^{0J}_j \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right) e^t_0 e^k_J e^j_K \\
\approx\; & \frac{1}{2\alpha_0}\Pi^k_{0I}\Pi^{0I}_j e^j_K + \frac{1}{4\alpha_0} e^k_K \Pi^i_{0I}\Pi^{0I}_i + 4\alpha_0 \left(e^m_K C^{kl}{}_{IJ} C_{ml}{}^{IJ} - \frac{1}{4} e^k_K C^{ij}{}_{IJ} C_{ij}{}^{IJ}\right) \\
& - 2\left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right)\left(R_{ij}{}^{IJ} e^i_I e^k_J e^j_K - \frac{1}{2} e^k_K R_{ij}{}^{IJ} e^i_I e^j_J\right) + \{p^k_K, \mathscr{H}_{H,\psi}\} - e^k_K \mathscr{H}_{H,\psi} \\
& - 2u^{0J}_j \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right) e^t_0 e^k_J e^j_K \ . \qquad \text{(B.0.13)}
\end{aligned}
$$

Contracting with e^K_k we obtain

$$
\begin{aligned}
0 \approx\; & \frac{5}{4\alpha_0}\Pi^k_{0I}\Pi^{0I}_k + \alpha_0 C_{ij}{}^{IJ}C^{ij}{}_{IJ} + \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right) R_{ij}{}^{IJ} e^i_I e^j_J \\
& + e^K_k\{p^k_K, \mathscr{H}_{H,\psi}\} - 3\mathscr{H}_{H,\psi} - 2u^{0J}_j \left(\frac{1}{\kappa^2} - \frac{H^2}{12}\right) e^t_0 e^j_J \ . \qquad \text{(B.0.14)}
\end{aligned}
$$

Combining Eqs. (B.0.14) and Eq. (B.0.12) we have a constraint equation

$$
\begin{aligned}
0 &\approx \frac{1}{2\alpha_0}\Pi^k_{0I}\Pi^{0I}_k + 2\alpha_0 C_{ij}{}^{lk}C^{ij}{}_{lk} + \frac{1}{4\alpha_0}\Pi^k_{IJ}\Pi^{IJ}_k + 4\alpha_0 C_{ij}{}^{0I}C^{ij}{}_{0I} \\
&\quad + e^K_k\{p^k_K, \mathscr{H}_{H,\psi}\} - e^0_t\{p^t_0, \mathscr{H}_{H,\psi}\} - 2\mathscr{H}_{H,\psi} \\
&\approx \frac{1}{2\alpha_0}\Pi^k_{0I}\Pi^{0I}_k + 2\alpha_0 C_{ij}{}^{lk}C^{ij}{}_{lk} + i\bar{\psi}\left(\gamma^I e^i_I D_i\psi - 2m\psi\right) - 2\mu^2 H^2 + 2\lambda H^4 \\
&=: \ \chi \ ,
\end{aligned}
\tag{B.0.15}
$$

which is not a linear combination of the primary constraints. In conclusion, $\chi = 0$ is a secondary constraint, which arises from the consistency condition.

Bibliography

[1] A. Connes, "On the spectral characterization of manifolds," *J. Noncom. Geom.*, vol. 7, p. 1, (2013).

[2] M. A. Rieffel, "Noncommutative tori: A case study of noncommutative differentiable manifolds," *Contemp. Math.*, vol. 105, pp. 191–211, (1990).

[3] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives.* Hindustan Book Agency, India, (2008).

[4] W. D. van Suijlekom, *Noncommutative geometry and particle physics.* Mathematical Physics Studies, Springer, New York, (2015).

[5] A. H. Chamseddine and A. Connes, "Resilience of the spectral standard model," *JHEP*, vol. 09, p. 104, (2012).

[6] C. A. Stephan, "New scalar fields in noncommutative geometry," *Phys. Rev. D*, vol. 79, p. 065013, (2009).

[7] A. H. Chamseddine, A. Connes, and W. D. van Suijlekom, "Beyond the spectral standard model: Emergence of pati-salam unification," *JHEP*, vol. 11, p. 132, (2013).

[8] A. Devastato, F. Lizzi, and P. Martinetti, "Grand symmetry, spectral action, and the higgs mass," *JHEP*, vol. 01, p. 042, (2014).

[9] L. Boyle and S. Farnworth, "A new algebraic structure in the standard model of particle physics," *arXiv: 1604.00847v1 [hep-th]*, (2016).

[10] A. H. Chamseddine, A. Connes, and V. Mukhanov, "Quanta of geometry: Noncommutative aspects," *Phys. Rev. Lett.*, vol. 114, p. 091302, (2015).

[11] J. Aastrup and J. M. Grimstrup, "Intersecting quantum gravity with noncommutative geometry - a review," *SIGMA*, vol. 8, p. 018, (2012).

[12] J. Aastrup and J. M. Grimstrup, "c^*-algebras of holonomy-diffeomorphisms and quantum gravity i," *Class. Quant. Grav.*, vol. 30, p. 085016, (2013).

[13] J. Aastrup and J. M. Grimstrup, "c^*-algebras of holonomy-diffeomorphisms and quantum gravity ii," *J. Geom. Phys.*, vol. 99, no. 10-19, (2016).

[14] P. Martinetti, "Line element in quantum gravity: the examples of dsr and noncommutative geometry," *Int. J. Mod. Phys. A*, vol. 32, pp. 2792–2801, (2009).

[15] J. Bellissard, A. van Elst, and H. Schulz-Baldes, "The non-commutative geometry of the quantum hall effect," *J. Math. Phys.*, vol. 35, p. 5373, (1994).

[16] A. Carey, K. Hannabuss, and V. Mathai, "Quantum hall effect and noncommutative geometry," *Journal of Geometry and Symmetry in Physics*, vol. 6, pp. 16–37, (2006).

[17] M. Kurkov, F. Lizzi, M. Sakellariadou, and A. Watcharangkool, "Spectral action with zeta function regularisation," *Phys. Rev. D*, vol. 91, p. 065013, (2015).

[18] M. Sakellariadou and A. Watcharangkool, "Linear stability of noncommutative spectral geometry," *Phys. Rev. D*, vol. 93, p. 064034, (2016).

[19] M. Sakellariadou and A. Watcharangkool, "Noncommutative geometrical origin of the energy-momentum dispersion relation," *Phys. Rev. D*, vol. 95, p. 025027, (2017).

[20] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa, *Element of noncommutative geometry.* Birkhäuser, Boston, (2001).

[21] A. Connes, *Noncommutative Geometry.* Academic Press, New York, (1994).

[22] J. C. Várilly, "An introduction to noncommutative geometry," *arXive:physics/9709045v1[math-ph]*, (1997).

[23] A. H. Chamseddine, A. Connes, and W. D. van Suijlekom, "Inner fluctuations in noncommutative geometry without the first order condition," *J. Geom. Phys.*, vol. 73, p. 222, (2013).

[24] A. H. Chamseddine and A. Connes, "Conceptual explanation for the algebra in the noncommutative approach to the standard model," *Phys. Rev. Lett.*, no. 99, p. 191601, (2007).

[25] L. Boyle and S. Farnworth, "The graded product of real spectral triples," *arXiv:1605.07035v1 [math-ph]*, (2016).

[26] P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Siner index theorem.* CRC Press, Boca Raton, FL, (1995).

[27] P. Martinetti and R. Wulkenhaar, "Discrete kaluza-klein from scalar fluctuations in noncommutative geometry," *J. Math. Phys.*, vol. 43, p. 182, (2002).

[28] A. Connes and A. H. Chamseddine, "Inner fluctuations of the spectral action," *J. Geom. Phys.*, vol. 57, p. 1, (2006).

[29] M. Eckstein and A. Zajac, "Asymptotic and exact expansions of heat traces," *Math. Phys. Anal. Geom.*, vol. 18, p. 28, (2015).

[30] S. Minakshisundaram and A. Pleijel, "Some properties of the eigenfunctions of the laplace operator on riemannian manifolds," *Can. J. Math.*, vol. 1, p. 242, (1949).

[31] A. H. Chamseddine, A. Connes, and M. Marcolli, "Gravity and the standard model with neutrino mixing," *Adv. Theor. Math. Phys.*, vol. 11, p. 991, (2017).

[32] B. Iochum, C. Levy, and D. Vassilevich, "Spectral action beyond the weak-field approximation," *Commun. Math. Phys.*, vol. 316, p. 595, (2012).

[33] M. A. Kurkov, F. Lizzi, and D. Vassilevich, "High energy bosons do not propagate," *Phys. Lett. B*, vol. 731, p. 311, (2014).

[34] W. D. van Suijlekom, "Renormalizability conditions for almost-commutative geometries," *Phys. Lett. B*, vol. 711, pp. 434–438, (2012).

[35] B. Iochum, C. Levy, and D. Vassilevich, "Global and local aspects of spectral actions," *J. Phys. A*, vol. 45, p. 374020, (2012).

[36] A. H. Chamseddine and A. Connes, "The spectral action principle," *Commun. Math. Phys.*, vol. 186, p. 731, (1997).

[37] S. W. Hawking, "Zeta function regularization of path integrals in curved space-time," *Commun. Math. Phys.*, vol. 55, p. 133, (1977).

[38] D. V. Vassilevich, "Heat kernel expansion: User's manual," *Phys. Rept.*, vol. 388, p. 279, (2003).

[39] A. A. Andrianov and F. Lizzi, "Bosonic spectral action induced from anomaly cancelation," *JHEP*, vol. 1005, p. 057, (2010).

[40] A. A. Andrianov, M. A. Kurkov, and F. Lizzi, "Spectral action, weyl anomaly and the higgs-dilaton potential," *JHEP*, vol. 1110, p. 001, (2011).

[41] M. A. Kurkov and F. Lizzi, "Higgs-dilaton lagrangian from spectral regularization," *Mod. Phys. Lett. A*, vol. 27, p. 1250203, (2012).

[42] M. A. Kurkov and M. Sakellariadou, "Spectral regularisation: Induced gravity and the onset of inflation," *JCAP* **1401** *(2014) 01, 035*, vol. 2014, no. 01, p. 035, (2014).

[43] L. S. Brown and J. C. Collins, "Dimensional renormalization of scalar field theory in curved space-time," *Annals Phys.*, vol. 130, p. 215, (1980).

[44] W. D. van Suijlekom, "Renormalization of the asymptotically expanded yang-mills spectral action," *Commun. Math. Phys.*, vol. 312, p. 883, (2012).

[45] A. A. Slavnov, "Invariant regularization of nonlinear chiral theories," *Nucl. Phys. B*, vol. 31, p. 301, (1971).

[46] C. P. Martin and F. R. Ruiz, “Higher covariant derivative regulators and nonmultiplicative renormalization,” *Phys. Lett. B*, vol. 343, p. 218, (1995).

[47] P. A. R. A. *et al.* [Planck Collaboration], “Planck 2013 results. xvi. cosmological parameters,” *Astron. Astrophys.*, (2014).

[48] K. A. O. *et al.* [Particle Data Group Collaboration], “Review of particle physics,” *Chin. Phys. C*, vol. 38, p. 090001, (2014).

[49] J. B. Fixler, G. T. Foster, J. M. McGuirk, and M. A. Kasevich, “Atom interferometer measurement of the newtonian constant of gravity,” *Science*, vol. 315, pp. 74–77, (2007).

[50] N. Alkofer, F. Saueressig, and O. Zanusso, “Spectral dimensions from the spectral action,” *Phys. Rev. D*, vol. 91, p. 025025, (2015).

[51] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space.* Cambridge University Press, (1982).

[52] K. S. Stelle, “Renormalization of higher derivative quantum gravity,” *Phys. Rev. D*, vol. 16, p. 953, (1977).

[53] G. Calcagni, L. Modesto, and G. Nardelli, “Quantum spectral dimension in quantum field theory,” *Int. J. Mod. Phys. D*, vol. 25, p. 1650058, (2016).

[54] T. J. Chen, M. Fasiello, E. A. Lim, and A. J. Tolley, “Higher derivative theories with constraints: Exorcising ostrogradski’s ghost,” *JCAP*, vol. 1302, p. 042, (2013).

[55] J. Z. Simon, “Higher-derivative lagrangians, nonlocality, problems, and solutions,” *Phys. Rev. D*, vol. 41, p. 3720, (1990).

[56] R. P. Woodard, "Ostrogradsky's theorem on hamiltonian instability," *Scholarpedia*, vol. 10, p. 32243, (2015).

[57] H. J. Schmidt, "Stability and hamiltonian formulation of higher derivative theories," *Phys. Rev. D*, vol. 49, p. 6354, (1994).

[58] O. Müller and M. Sánchez, "An invitation to lorentzian geometry," *Jahresber. Deutsch. Math.-Verein.*, vol. 115, p. 153, (2013).

[59] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems.* Princeton University Press, Princeton, NJ, (1992).

[60] J. T. Wheeler, "Weyl gravity as general relativity," *Phys. Rev. D*, vol. 90, p. 025027, (2014).

[61] F. Hanisch, F. Pfaeffle, and C. A. Stephan, "The spectral action for dirac operators with skew-symmetric torsion," *Commun. Math. Phys.*, vol. 300, p. 877, (2010).

[62] F. Pfaeffle and C. A. Stephan, "On gravity, torsion and the spectral action principle," *J. Functional Analysis*, vol. 262, p. 1529, (2012).

[63] J. T. Trujillo, *Weyl gravity as a gauge theory.* Ph.d. thesis, Utah State University, http://digitalcommons.usu.edu/etd/1951, (2013).

[64] P. Szekeres, "Spaces conformal to a class of spaces in general relativity," *Proc. R. Soc. A*, vol. 274, p. 206, (1963).

[65] M. Han, Y. Ma, Y. Ding, and L. Qin, "Hamiltonian analysis of n-dimensional palatini gravity with matter," *Mod. Phys. Lett. A*, vol. 20, p. 725, (2005).

[66] N. Franco and M. Eckstein, "Causality in noncommutative two-sheeted space-times," *J. Geom. Phys.*, vol. 96, p. 42, (2015).

[67] K. van den Dungen, "Krein spectral triples and the fermionic action," *Math. Phys. Anal. Geom.*, vol. 19, p. 4, (2015).

[68] A. Strohmaier, "On noncommutative and semi-riemannian geometry," *J. Geom. Phys.*, vol. 56, pp. 175–195, (2006).

[69] K. van den Dungen, M. Paschke, and A. Rennie, "Pseudo-riemannian spectral triples and the harmonic oscillator," *J. Geom. Phys.*, vol. 73, pp. 37–55, (2013).

[70] N. Franco and M. Eckstein, "An algebraic formulation of causality for noncommutative geometry," *Class. Quant. Grav.*, vol. 30, p. 135007, (2013).

[71] N. Franco and M. Eckstein, *MATHEMATICAL STRUCTURES OF THE UNIVERSE: Noncommutative geometry, Lorentzian structures and causality.* Copernicus Center Press, (2014).

[72] N. Franco, "Global eikonal condition for lorentzian distance function in noncommutative geometry," *SIGMA*, vol. 6, p. 064, (2010).

[73] B. Iochum, C. Levy, and D. Vassilevich, "Spectral action for torsion with and without boundaries," *Commun. Math. Phys.*, vol. 310, pp. 367 – 382, (2012).